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Nested Bethe ansatz for "all" closed spin chains

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Abstract

We present in an unified and detailed way the Nested Bethe Ansatz for closed spin chains based on $\mathcal{Y}(gl(\mathfrak{n}))$, $\mathcal{Y}(gl(\mathfrak{m}|\mathfrak{n}))$, $\widehat{\mathcal{U}}_q(gl(\mathfrak{n}))$ or $\widehat{\mathcal{U}}_q(gl(\mathfrak{m}|\mathfrak{n}))$ (super)algebras, with arbitrary representations (i.e. 'spins') on each site of the chain. In particular, the case of indecomposable representations of superalgebras is studied. The construction extends and unifies the results already obtained for spin chains based on $\mathcal{Y}(gl(\mathfrak{n}))$ or $\widehat{\mathcal{U}}_q(gl(\mathfrak{n}))$ and for some particular super-spin chains. We give the Bethe equations and the form of the Bethe vectors. The case of $gl(2|1)$, $gl(2|2)$ and $gl(4|4)$ superalgebras (that are related to AdS/CFT correspondence) is also detailed.

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1 Introduction

Finding eigenvectors and eigenvalues of a transfer matrix is a fundamental problem in integrable systems. It started with the work of Bethe, which led to the celebrated Bethe ansatz [1]. Then, the framework of the Quantum Inverse Scattering Problem based on the Yang-Baxter equation became one of the most used way to adress this question. This technique is being developed since the 70's by the Leningrad School, see for example the review [2] and references therein. Since then, numerous publications have been devoted to the subject, so that it is becoming difficult to make exhaustive citation. In the seek of such an impossible task, we will focus on Bethe equations and Bethe vectors for closed (or periodic) spin chains based on $gl(\mathfrak{n})$ or $gl(\mathfrak{m}|\mathfrak{n})$ algebras (leading to generalized XXX (super)spin chains) and their quantum deformations (leading to generalized XXZ (super)spin chains). The resolution of the general spin chain model started with the calculation of the Bethe equations, computed for $gl(\mathfrak{n})$ chains (with spins in the fundamental representation) in [3, 4]. Other cases (e.g. combining different spins) have been done in [5], see also [6, 7]. Closed spin chains based on $gl(\mathfrak{m}|\mathfrak{n})$ superalgebras in the distinguished diagram were studied in [8] and [9] and, in the case of alternating fundamental-conjugate representations of $gl(\mathfrak{m}|\mathfrak{n})$ in [10]. In [11], closed spin chains in the fundamental representation but for any type of Dynkin diagram where studied using the Baxter Q-operator, and generalized in [12] to a chain where all the spins are in a (type 1) typical representation depending on a free parameter. The general approach for arbitrary representations with any type of Dynkin diagram was done in [13]. General approach using Hirota equation was done in [14]. The case of quantum deformations was dealt in [15, 16] for algebras (see also [17] for a global treatment)

However, in most of the above papers, one computes the Bethe equations and the transfer matrix eigenvalues, but not the Bethe vectors (i.e. the transfer matrix eigenvectors). To get them, one needs a more involved Bethe ansatz, the algebraic Bethe ansatz [18] (for rank 1 algebras) and its refinement to higher rank algebras, the nested Bethe ansatz [19, 20]. Algebraic Bethe ansatz for a general $gl(2)$ spin chain can be found in [21]. Generalization to superalgebras has been done in some particular cases, such as the $gl(1|2)$ superalgebra [22]. Nested Bethe ansatz for generalized XXZ spin chains with fundamental representations has been studied [23]. Alternating generalized XXZ super-spin chain has been treated in [24].

More recently, a unified presentation for Bethe vectors of $gl(\mathfrak{n})$ and $\mathcal{U}_q(gl(\mathfrak{n}))$ spin chains has been developped [25, 26], producing a 'trace formula' for Bethe vectors. This trace formula was shown to obey the same recursion formula that is obtained from the nested Bethe ansatz, proving equivalence between the two approaches.

Let us also mention an alternative approach [27, 28] to the construction of Bethe vectors, using currents in the Drinfeld presentation of (quantum) algebras. The construction is off-shell (i.e. without any reference to Bethe equations) and thus may open a way to compute correlation functions. In this formalism, the construction is done without any reference to a highest weight, but rather computing modulo a suitably defined Borel subalgebra. The Bethe vectors are then viewed as special projections of currents that obey (Bethe ansatz) comultiplication properties. Note that these properties are valid even without Bethe equations: these equations appear when asking the off-shell Bethe vectors to be eigenvectors of the transfer matrix [29]. The construction (and the connection with the previous approach) has been done for $\mathcal{Y}(\mathfrak{n})$ and $\widehat{\mathcal{U}}_q(\mathfrak{n})$ algebras [30]. The case of (deformed) superalgebras remains to be treated in this formalism.

In the present paper we present in a unified way the nested Bethe ansatz for spin chains based on $gl(\mathfrak{n})$, $gl(\mathfrak{m}|\mathfrak{n})$, $\mathcal{U}_q(gl(\mathfrak{n}))$ and $\mathcal{U}_q(gl(\mathfrak{m}|\mathfrak{n}))$ (super)algebras, with arbitrary representations (i.e. 'spins') on each site of the chain. In the case of (quantum) algebras, the construction is equivalent to the 'trace formula' approach, and we make contact between the two presentations. Our construction also works for (quantum) superalgebras and we exhibit a 'supertrace formula' for the Bethe vectors. The technique is essentially algebraic and works as soon as the representations on the spin chains are highest weight. Then we use the Shapovalov form to prove orthogonality condition between Bethe vectors.

The plan of the paper is as follows. In section 2, we introduce the different algebras that are concerned with our approach, presenting their R-matrices and their finite-dimensional irreducible representations. In section 3, as a warm up, we remind the algebraic Bethe ansatz, which deals with spin chains based on $gl(2)$, $gl(1|1)$ algebras and their quantum deformations. Then, in section 4, we perform the nested Bethe ansatz in a very detailled and pedestrian way[4] and up to the end. Finally, in section 5, we study the Bethe vectors that have been constructed in the prevous section, showing connection with the 'trace formula' and generalizing it to (quantum) superalgebras. Some examples of Bethe vectors are also given. The case of $gl(2|1)$, $gl(2|2)$ and $gl(4|4)$ superalgebras (that are related to AdS/CFT correspondence) is detailed in section 6. An appendix is devoted to the presentations of the finite dimensional (super)algebras used in the paper.

2 Algebraic structures for closed spin chains

2.1 Auxiliary graded spaces

We use the so-called auxiliary space framework, a useful notation for the R-matrix formalism. In this formalism, one deals with multiple tensor product of vectorial spaces $\mathcal{V} \otimes \cdots \otimes \mathcal{V}$ and operators (defining an algebra $\mathcal{A}$) therein. For a matrix valued operator $A := \sum_{ij} E_{ij} \otimes A_{ij} \in End(\mathcal{V}) \otimes \mathcal{A}$, and any numbers $k \leq m$ we set:

$$A_k := \sum_{ij} \mathbb{I}^{\otimes(k-1)} \otimes E_{ij} \otimes \mathbb{I}^{\otimes(m-k)} \otimes A_{ij} \in End(\mathcal{V}^{\otimes m}) \otimes \mathcal{A} \,, \ 1 \leq k \leq m \qquad (2.1)$$

where E_{ij} are elementary matrices with 1 at position (i,j) and 0 elsewhere.

The notation is also valid for complex matrices, taking $\mathcal{A} := \mathbb{C}$ and using the isomorphism $End(\mathcal{V}) \otimes \mathbb{C} \sim End(\mathcal{V})$.

When $A \in End(\mathcal{V}) \otimes End(\mathcal{V}) \otimes \mathcal{A}$, for k, l such that $1 \leq k < l \leq m$, we denote by A_{kl} the operator in $End(\mathcal{V}^{\otimes m}) \otimes \mathcal{A}$ defined by

$$A_{kl} := \sum_{ijab} \mathbb{I}^{\otimes(k-1)} \otimes E_{ij} \otimes \mathbb{I}^{\otimes(l-k-1)} \otimes E_{ab} \otimes \mathbb{I}^{\otimes(m-l)} \otimes A_{ijab} \,. \qquad (2.2)$$

We will work on $\mathbb{Z}_2$-graded spaces $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$. The elementary $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ column vectors e_i (with 1 at position i and 0 elsewhere) and elementary $End(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ matrices E_{ij} have grade

$$[e_i] = [i] \quad \text{and} \quad [E_{ij}] = [i] + [j]. \qquad (2.3)$$

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[4] At least to our opinion. . .

This grading is also extended to the superalgebras we deal with, see section 2.3 below.

The tensor product is graded accordingly:

$$(A_{ij} \otimes A_{kl})(A_{ab} \otimes A_{cd}) = (-1)^{([k]+[l])([a]+[b])}(A_{ij}A_{ab} \otimes A_{kl}A_{cd}). \tag{2.4}$$

To simplify the presentation we work with the *distinguished* $\mathbb{Z}_2$-grade defined by

$$[i] = \begin{cases} 0, & 1 \le i \le \mathfrak{m}, \\ 1, & \mathfrak{m}+1 \le i \le \mathfrak{m}+\mathfrak{n}. \end{cases} \tag{2.5}$$

Simplification in the expressions follows from the following rule:

$$[i][j] = [i] \quad \text{when} \quad i \le j, \tag{2.6}$$

which is valid only for the distinguished grade. Generalization to other gradings is easy to do.

The non graded case is obtained setting formally $\mathfrak{n} = 0$ in the above expressions.

2.2 R matrices

In what follows, we will deal with different types of $R \in End(\mathcal{V}) \otimes End(\mathcal{V})$ matrices, all obeying (graded) Yang-Baxter equation (writen in auxiliary space $End(\mathcal{V}) \otimes End(\mathcal{V}) \otimes End(\mathcal{V})$):

$$R_{12}(u_1, u_2)\, R_{13}(u_1, u_3)\, R_{23}(u_2, u_3) \quad = \quad R_{23}(u_2, u_3)\, R_{13}(u_1, u_3)\, R_{12}(u_1, u_2) \tag{2.7}$$

and unitarity relation:

$$R_{12}(u, v)R_{21}(v, u) \quad = \quad \zeta(u, v)\, \mathbb{I} \otimes \mathbb{I}, \tag{2.8}$$

where $\zeta(u, v)$ is a $\mathbb{C}$-function depending on the model under consideration (see below). These are the two fondamental properties used to construct transfer matrix for periodic spin chains. Below, we focus on infinite dimensional associatives algebras based on $gl(\mathfrak{n})$ and $gl(\mathfrak{m}|\mathfrak{n})$ Lie (super) algebras: Yangians $\mathcal{Y}(gl(\mathfrak{n})) \equiv \mathcal{Y}(\mathfrak{n})$, super Yangians $\mathcal{Y}(gl(\mathfrak{m}|\mathfrak{n})) \equiv \mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, quantum affine (super) group $\mathcal{U}_q(\widehat{gl}(\mathfrak{n})) \equiv \widehat{\mathcal{U}}_q(\mathfrak{n})$ and $\mathcal{U}_q(\widehat{gl}(\mathfrak{m}|\mathfrak{n})) \equiv \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$. We note these algebras $\mathcal{A}_\mathfrak{n} = Y(\mathfrak{n})$ or $\widehat{\mathcal{U}}_q(\mathfrak{n})$ and $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}} = \mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ or $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$. As a notation, we will write also $\mathcal{A}_{\mathfrak{m}|0} = \mathcal{A}_\mathfrak{m}$.

Depending on the choice of the algebra, we will construct different spin chains:

- **For $gl(\mathfrak{n})$ or generalized XXX spin chains,** the algebra is the Yangian $\mathcal{Y}(\mathfrak{n})$ with rational R-matrix:

$$R_{12}(u, v) \quad = \quad R_{12}(u-v) = (u-v)\, \mathbb{I} \otimes \mathbb{I} - \hbar\, P_{12} \quad \text{with} \quad P_{12} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \tag{2.9}$$

  where u is a spectral parameter over the field $\mathbb{C}$ and P_{12} is the permutation operator $(P_{12}(a \otimes b) = b \otimes a)$. It is the simplest rational solution of the Yang-Baxter equation found by Yang and Baxter in [31, 32] and studied by Drinfel'd [33, 34] in connection with enveloping Lie algebras. When $\mathfrak{n} = 2$ and all the spins are in fundamental (i.e. spin $\frac{1}{2}$) representation, the spin chain model constructed from this R-matrix is the celebrated Heisenberg XXX model [35].

3

The unitarity relation reads:

$$R_{12}(u,v)\, R_{21}(v,u) \;=\; (u-v-\hbar)(v-u-\hbar)\,\mathbb{I}\otimes\mathbb{I}\,. \tag{2.10}$$

Note that the matrix is symmetric:

$$R_{21}(u) \equiv P_{12}\, R_{12}(u)\, P_{12} = R_{12}(u)\,. \tag{2.11}$$

From the mathematical point of view, the parameter of deformation $\hbar$ is irrelevent since we have the isomorphism $\mathcal{Y}_\hbar(\mathfrak{n}) \sim \mathcal{Y}_{\hbar'}(\mathfrak{n})$ for any non-vanishing values of $\hbar$ and $\hbar'$. For this reason, it is in general set to 1 in the mathematical litterature. However, in spin chains studies, it is set to $\pm i$, to ensure that the Hamiltonian is Hermitian. In this paper, we keep it free to encompass these two conventions.

- **For $gl(\mathfrak{m}|\mathfrak{n})$ or supersymmetric XXX spin chains,** one considers the super-Yangian $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, introduced in [19, 8, 36] with the same form (2.9) for the R-matrix and unitarity relation (2.10), but with a $\mathbb{Z}_2$ graded auxiliary space. The permutation operator in the graded space takes the form:

$$P_{12} = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[j]} E_{ij}\otimes E_{ji}\,, \tag{2.12}$$

so that we have $P_{12}(a\otimes b) = (-1)^{[a][b]}\, b\otimes a$.

- **We will also deal with $\widehat{\mathcal{U}}_q(\mathfrak{n})$ or generalized XXZ spin chains.** In that case, one considers the R-matrix of the (centerless) affine quantized algebra $\widehat{\mathcal{U}}_q(\mathfrak{n})$:

$$
\begin{aligned}
R_{12}(u,v) \;=\; R_{12}\Big(\frac{u}{v}\Big) &= \Big(\frac{u\,q}{v}-\frac{v}{u\,q}\Big)\sum_{a=1}^{\mathfrak{n}} E_{aa}\otimes E_{aa} + \Big(\frac{u}{v}-\frac{v}{u}\Big)\sum_{1\le a\ne b\le \mathfrak{n}} E_{aa}\otimes E_{bb} \\
&\quad +(q-q^{-1})\sum_{1\le a\ne b\le \mathfrak{n}} \Big(\frac{u}{v}\Big)^{sign(b-a)} E_{ab}\otimes E_{ba}
\end{aligned}
\tag{2.13}
$$

with unitarity relation:

$$R_{12}(u,v)\, R_{21}(v,u) \;=\; \Big(\frac{u\,q}{v}-\frac{v}{u\,q}\Big)\Big(\frac{v\,q}{u}-\frac{u}{v\,q}\Big)\,\mathbb{I}\otimes\mathbb{I} \tag{2.14}$$

where q is a generic complex number, not root of unity. It has been introduced by Jimbo or Faddeev, Reshetikhin and Takhtajan [15, 37]. When $\mathfrak{n}=2$ and the spins lie in fundamental representation we recover the Heisenberg XXZ model.

- **The last cases considered are $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$ or supersymmetric XXZ spin chains.** The R-matrix of the (centerless) affine quantized algebra $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$ reads [38, 39, 40]:

$$
\begin{aligned}
R_{12}(u,v) \;=\; R_{12}\Big(\frac{u}{v}\Big) &= \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}}\Big(\frac{u}{v}q^{1-2[a]}-\frac{v}{u}q^{-1+2[a]}\Big) E_{aa}\otimes E_{aa} + \Big(\frac{u}{v}-\frac{v}{u}\Big)\sum_{1\le a\ne b\le \mathfrak{m}+\mathfrak{n}} E_{aa}\otimes E_{bb} \\
&\quad +(q-q^{-1})\sum_{1\le a\ne b\le \mathfrak{m}+\mathfrak{n}} \Big(\frac{u}{v}\Big)^{sign(b-a)}(-1)^{[b]} E_{ab}\otimes E_{ba}\,.
\end{aligned}
\tag{2.15}
$$

4

The auxiliary space is graded, and the unitarity relation reads:

$$R_{12}(u,v)\,R_{21}(v,u) \;=\; (\frac{u\,q}{v} - \frac{v}{u\,q})(\frac{v\,q}{u} - \frac{u}{v\,q})\,\mathbb{I}\otimes\mathbb{I}\,. \tag{2.16}$$

- **We will encompass all these cases writing:**

$$R_{12}(u,v) \;=\; \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{a}_a(u,v)E_{aa}\otimes E_{aa} + \mathfrak{b}(u,v)\sum_{1\le a\neq b\le \mathfrak{m}+\mathfrak{n}} E_{aa}\otimes E_{bb}$$
$$+ \sum_{1\le a\neq b\le \mathfrak{m}+\mathfrak{n}} \mathfrak{c}_{ab}(u,v)E_{ab}\otimes E_{ba} \tag{2.17}$$

with the following identifications:

$$\text{For} \quad Y(\mathfrak{n}): \qquad \mathfrak{a}_a(u,v) = u - v - \hbar \;\; ; \;\; \mathfrak{b}(u,v) = u - v \;\; ; \;\; \mathfrak{c}_{ab}(u,v) = -\hbar \quad (2.18)$$

$$\text{For} \quad Y(\mathfrak{m}|\mathfrak{n}): \qquad \mathfrak{a}_a(u,v) = u - v - (-1)^{[a]}\hbar \;\; ; \;\; \mathfrak{b}(u,v) = u - v$$
$$\mathfrak{c}_{ab}(u,v) = -(-1)^{[b]}\hbar \tag{2.19}$$

$$\text{For} \quad \widehat{\mathcal{U}}_q(\mathfrak{n}): \qquad \mathfrak{a}_a(u,v) = (\frac{u\,q}{v} - \frac{v}{u\,q}) \;\; ; \;\; \mathfrak{b}(u,v) = \frac{u}{v} - \frac{v}{u}$$
$$\mathfrak{c}_{ab}(u,v) = (q - q^{-1})(\frac{u}{v})^{sign(b-a)} \tag{2.20}$$

$$\text{For} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}): \qquad \mathfrak{a}_a(u,v) = (\frac{u}{v}q^{1-2[a]} - \frac{v}{u}q^{-1+2[a]}) \;\; ; \;\; \mathfrak{b}(u,v) = \frac{u}{v} - \frac{v}{u}$$
$$\mathfrak{c}_{ab}(u,v) = (q - q^{-1})(\frac{u}{v})^{sign(b-a)}(-1)^{[b]} \tag{2.21}$$

In this notation, the unitary relation reads

$$\zeta(u,v) = \mathfrak{a}_1(u,v)\mathfrak{a}_1(v,u)\,. \tag{2.22}$$

Remark that we have the properties

$$\mathfrak{a}_k(u,v)\,\mathfrak{a}_k(v,u) = \mathfrak{a}_l(u,v)\,\mathfrak{a}_l(v,u)\,, \; \forall \, k,l \tag{2.23}$$
$$\mathfrak{b}(u,v) = -\mathfrak{b}(v,u) \quad \text{and} \quad \mathfrak{c}_{ab}(u,v) = (-1)^{[a]+[b]}\,\mathfrak{c}_{ba}(v,u) \tag{2.24}$$

We will also use 'reduced' R-matrices $R^{(k)}(u)$, deduced from $R(u)$ by suppressing all the terms containing indices j with $j < k$:

$$R^{(k)}_{12}(u,v) \;=\; \sum_{a=k}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{a}_a(u,v)E_{aa}\otimes E_{aa} + \mathfrak{b}(u,v)\sum_{k\le a\neq b\le \mathfrak{m}+\mathfrak{n}} E_{aa}\otimes E_{bb}$$
$$+ \sum_{k\le a\neq b\le \mathfrak{m}+\mathfrak{n}} \mathfrak{c}_{ab}(u,v)E_{ab}\otimes E_{ba}\,. \tag{2.25}$$

Hence, we have $R^{(1)}_{12}(u,v) = R_{12}(u,v)$, and more generally $R^{(k)}_{12}(u,v)$ corresponds to the embbedding $\mathcal{A}_{\mathfrak{m}+1-k|\mathfrak{n}} \subset \mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ when $k \le \mathfrak{m}+1$ or $\mathcal{A}_{0|\mathfrak{n}-(k-\mathfrak{m}-1)} \subset \mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ otherwise. In the following, to make the presentation concise, we will write, for a generic k, $\mathcal{A}_{\mathfrak{m}+1-k|\mathfrak{n}}$, keeping in mind that one should write $\mathcal{A}_{0|\mathfrak{n}-(k-\mathfrak{m}-1)}$ when $k > \mathfrak{m}+1$.

We define the normalized reduced R-matrices

$$\mathbb{R}^{(k)}_{12}(u,v) = \frac{1}{\mathfrak{a}_k(u,v)}\,R^{(k)}_{12}(u,v) \quad \text{such that} \quad \mathbb{R}^{(k)}_{12}(u,v)\,\mathbb{R}^{(k)}_{21}(v,u) = \mathbb{I}\otimes\mathbb{I}\,. \tag{2.26}$$

2.3 RTT relation and transfer matrix

The algebraic structures associated to spin chains are defined using the RTT relations [34, 37]. They allow to generate all the relations between each generator of the graded unital associative algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. We gather the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ generators into a $(\mathfrak{m} + \mathfrak{n}) \times (\mathfrak{m} + \mathfrak{n})$ matrix acting in an auxiliary space $\mathcal{V} = \mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ whose entries are formal series of a complex parameter u,

$$T(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t_{ij}(u) \in \mathcal{V} \otimes \mathcal{A}[[u, u^{-1}]] \,.$$

Since the auxiliary space $End(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ is interpreted as a representation of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ (see below), the $\mathbb{Z}_2$-grading of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ must correspond to the one defined on $End(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ matrices (section 2.1). Hence, the generator $t_{ij}(u)$ has grade $[i] + [j]$, so that the monodromy matrix $T(u)$ is globally even. As for matrices, the tensor product of algebras will be graded, as well as between algebras and matrices, e.g.

$$\left(E_{ij} \otimes t_{ij}(u)\right) \left(E_{kl} \otimes t_{kl}(u)\right) = (-1)^{([i]+[j])([k]+[l])} E_{ij} E_{kl} \otimes t_{ij}(u) \, t_{kl}(u) \,. \tag{2.27}$$

The 'true' generators $t_{ij}^{(n)}$ of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ appear upon expansion of $t_{ij}(u)$ in u. For the (super) Yangians $\mathcal{Y}(\mathfrak{n})$ and $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, $t_{ij}(u)$ is a series in u^{-1}:

$$t_{ij}(u) = \sum_{n=0}^{\infty} t_{ij}^{(n)} u^{-n} \quad \text{with} \quad t_{ij}^{(0)} = \delta_{ij} \,. \tag{2.28}$$

In the quantum affine (super)algebra case, a complete description of the algebras requires the introduction of two matrices $L^{\pm}(u)$

$$L^{\pm}(u) \;=\; \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes L_{ij}^{\pm}(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes \sum_{n=0}^{\infty} L_{ij}^{\pm(n)} u^{\pm n} \tag{2.29}$$

with relations:

$$L_{ii}^{+(0)} L_{ii}^{-(0)} = 1 \,, \forall i \quad \text{and} \quad L_{ij}^{+(0)} = 0 = L_{ji}^{-(0)} \,, \ i < j \tag{2.30}$$

$$R_{12}(u,v) \, L_1^{\pm}(u) \, L_2^{\pm}(v) \;=\; L_2^{\pm}(v) \, L_1^{\pm}(u) \, R_{12}(u,v) \,, \tag{2.31}$$

$$R_{12}(u,v) \, L_1^{\mp}(u) \, L_2^{\pm}(v) \;=\; L_2^{\pm}(v) \, L_1^{\mp}(u) \, R_{12}(u,v) \,. \tag{2.32}$$

However, in the context of evaluation representations it is sufficient to consider only one, say $T(u) = L^+(u) - L^-(u)$, to construct a transfer matrix, see e.g. [17] for more details.

Then, the RTT relations take the form:

$$R_{12}(u,v) \, T_1(u) \, T_2(v) = T_2(v) \, T_1(u) \, R_{12}(u,v) \,. \tag{2.33}$$

From the R-matrix (2.17), we get the commutation relations through an expansion on the graded basis $E_{ij} \otimes E_{kl}$:

$$\begin{aligned}
\mathfrak{b}(u,v) \, [t_{ij}(u) \,, t_{kl}(v)\} \;=\;\; & \delta_{ik} \left(\mathfrak{b}(u,v) - \mathfrak{a}_i(u,v)\right) t_{kj}(u) \, t_{il}(v) \\
& -(1 - \delta_{ik}) \, (-1)^{([i]+[k]) \, ([k]+[l])} \, \mathfrak{c}_{ik}(u,v) \, t_{kj}(u) \, t_{il}(v) \\
& -\delta_{jl} \, (-1)^{([k]+[l])([i]+[j])} \left(\mathfrak{b}(u,v) - \mathfrak{a}_j(u,v)\right) t_{kj}(v) \, t_{il}(u) \\
& +(1 - \delta_{jl}) \, (-1)^{[j]+[i][k]+[l] \, ([i]+[k])} \, \mathfrak{c}_{lj}(u,v) \, t_{kj}(v) \, t_{il}(u) \tag{2.34}
\end{aligned}$$

with

$$[t_{ij}(u)\,,\,t_{kl}(v)\} = t_{ij}(u)\,t_{kl}(v) - (-1)^{([i]+[j])([k]+[l])}\,t_{kl}(v)\,t_{ij}(u) \quad (2.35)$$

$$= -(-1)^{([i]+[j])([k]+[l])}\,[t_{kl}(v)\,,\,t_{ij}(u)\} \, . \quad (2.36)$$

In the context of spin chain models, $T(u)$ is called the (algebraic) monodromy matrix. The connection with usual monodromy matrix is done upon representation (see next section).

$\mathcal{A}$ has a Hopf algebra structure, whose coproduct Δ reads:

$$\Delta : \quad \begin{aligned} End(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}) \otimes \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \quad &\rightarrow \quad End(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}) \otimes \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \otimes \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \\ T(u) \quad &\mapsto \quad T(u)\dot{\otimes}T(u) = \sum_{i,j,k=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([k]+[i])([k]+[j])}\,E_{ij} \otimes t_{ik}(u) \otimes t_{kj}(u) \end{aligned} \quad (2.37)$$

More generally, one defines recursively for $L \geq 2$

$$\Delta^{(L+1)} = (\mathbb{I}^{\otimes(L-1)} \otimes \Delta) \circ \Delta^{(L)} \quad : \quad \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \rightarrow \mathcal{A}_{\mathfrak{m}|\mathfrak{n}}^{\otimes(L+1)} \quad (2.38)$$

with $\Delta^{(2)} = \Delta$ and $\Delta^{(1)} = \mathbb{I}$. The map $\Delta^{(L)}$ is an algebra homomorphism.

One defines the transfer matrix as the supertrace over the auxiliary space of the monodromy matrix:

$$t(u) = str\big(T(u)\big) = \sum_{i=1}^{\mathfrak{m}+\mathfrak{n}}(-1)^{[i]}t_{ii}(u) \, . \quad (2.39)$$

Relations (2.33) and (2.8) then show that the transfer matrices at two different values of the spectral parameter commute

$$[t(u), t(v)] = 0 \, . \quad (2.40)$$

Thus, $t(u)$ generates (via an expansion in u) a set of L (the number of sites) independent integrals of motion or charges in involution which ensure integrability of the model.

The diagonalisation of the transfer matrix can be done in an algebraic way when working in a highest weight representation. Thus, we briefly describe the representation theory of the algebras we use.

2.4   Finite dimensional representations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ and spin chains

The fundamental point in using the ABA is to know a pseudo-vacuum for the model. In the mathematical framework it is equivalent to know a highest weight vector for the representation of the algebra which underlies the model. We describe the link between highest weight vector of the standard finite dimensional Lie (super)algebras $gl(\mathfrak{n})$ or $gl(\mathfrak{m}|\mathfrak{n})$ and the infinite dimensional (graded) algebras $\mathcal{A}_{\mathfrak{n}}$ or $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$.

Definition 2.1 *A representation of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ is called highest weight if there exists a nonzero vector Ω such that:*

$$t_{ii}(u)\,\Omega = \Lambda_i(u)\,\Omega \quad and \quad t_{ij}(u)\,\Omega = 0 \ if \ i > j \quad (2.41)$$

for some scalars $\Lambda_i(u) \in \mathbb{C}\,[[u^{-1}]]$. $\Lambda(u) = (\Lambda_1(u), \ldots, \Lambda_{\mathfrak{m}+\mathfrak{n}}(u))$ is called the highest weight and Ω the highest weight vector.

The action of the T-matrix on Ω gives a triangular matrix. We can interpret the operators $t_{ij}(u)$ for $i \neq j$ as creation or annihilation operators. The main theorem on highest weights is:

Theorem 2.2 *Every finite-dimensional irreducible representation of $\mathcal{A}_{\mathfrak{n}}$ or $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ is highest weight. Moreover, it contains a unique (up to scalar multiples) highest weight vector.*

This theorem is presented in [33] for $\mathcal{Y}(\mathfrak{n})$, [41] for $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, [42] for $\widehat{\mathcal{U}}_q(n)$ and [43] for $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$.

To construct such representations, one uses the evaluation morphism, which relates the infinite dimensional algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ to its finite dimensional subalgebra $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$. The correspondence between the algebras $\mathcal{A}$ and $\mathcal{B}$ is given in table (2.42). The algebraic structure of the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ algebras and their irreducible finite dimensional representations are described in the appendix.

(Super)algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$	$\mathcal{Y}(\mathfrak{n})$	$\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$	$\widehat{\mathcal{U}}_q(n)$	$\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$
Subalgebra $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$	$gl(\mathfrak{n})$	$gl(\mathfrak{m}|\mathfrak{n})$	$\mathcal{U}_q(\mathfrak{n})$	$\mathcal{U}_q(\mathfrak{m}|\mathfrak{n})$

$$(2.42)$$

The evaluation morphism with parameter $a \in \mathbb{C}$ is given by

$$
ev_a : \begin{cases} \mathcal{V} \otimes \mathcal{A} & \to \quad \mathcal{V} \otimes \mathcal{B} \\ \\ T(u) & \mapsto \quad \begin{cases} (u-a)\mathbb{I} - \hbar\,\mathbb{E} & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ \dfrac{u}{a}\,L^+ - \dfrac{a}{u}\,L^- & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}) \end{cases} \end{cases}
\tag{2.43}
$$

where

$$
\mathbb{E} \;=\; \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[j]} E_{ij} \otimes \mathcal{E}_{ji} \in \mathcal{V} \otimes gl(\mathfrak{m}|\mathfrak{n}) \quad\text{and}\quad L^{\pm} = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes l_{ij}^{\pm} \in \mathcal{V} \otimes \mathcal{U}_q(\mathfrak{m}|\mathfrak{n}) \tag{2.44}
$$

with the convention $gl(\mathfrak{m}|0) \equiv gl(\mathfrak{m})$ and $\mathcal{U}_q(\mathfrak{m}|0) \equiv \mathcal{U}_q(\mathfrak{m})$ as for the infinite dimensional superalgebras $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ and $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$.

From the evaluation morphism ev_a and a highest weight representation π_λ of $\mathcal{B}$, one can construct a highest weight representation of $\mathcal{A}$, called evaluation representation:

$$
\rho_a^\lambda = ev_a \circ \pi_\lambda \;:\; \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \xrightarrow{\;ev_a\;} \mathcal{B}_{\mathfrak{m}|\mathfrak{n}} \xrightarrow{\;\pi_\lambda\;} \mathcal{V}_\lambda .
\tag{2.45}
$$

The weight of this evaluation representation is given by $\Lambda(u) = \big(\Lambda_1(u), \ldots, \Lambda_{\mathfrak{m}+\mathfrak{n}}(u)\big)$, with

$$
\Lambda_j(u) = \begin{cases} u - a - (-1)^{[j]}\,\hbar\,\lambda_j & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ (-1)^{[j]} \left(\dfrac{u}{a}\,\eta_j q^{\lambda_j} - \dfrac{a}{u}\,\eta_j q^{-\lambda_j} \right) & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}) \end{cases} \qquad j = 1, \ldots, \mathfrak{m}+\mathfrak{n}
\tag{2.46}
$$

where λ_j, $j = 1, \ldots, \mathfrak{m}+\mathfrak{n}$ are the weights of the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ representation (see appendix).

More generally, one constructs tensor product of evaluation representations using the coproduct of $\mathcal{A}$:

$$
\left(\otimes_{i=1}^{L}\, \rho_{a_i}^{\lambda^{\langle i \rangle}} \right) \circ \Delta^{(L)}\big(T(u)\big) = \rho_{a_1}^{\lambda^{\langle 1 \rangle}}\big(T(u)\big) \dot\otimes \rho_{a_2}^{\lambda^{\langle 2 \rangle}}\big(T(u)\big) \dot\otimes \cdots \dot\otimes \rho_{a_L}^{\lambda^{\langle L \rangle}}\big(T(u)\big)
\tag{2.47}
$$

8

where $\lambda^{\langle i\rangle} = (\lambda_1^{\langle i\rangle}, \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}}^{\langle i\rangle})$, $i = 1, \ldots, L$ are the weights of the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ representations. This provides a $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ representation with weight

$$\Lambda_j(u) = \prod_{i=1}^{L} \Lambda_j^{\langle i\rangle}(u), \qquad j = 1, \ldots, \mathfrak{m} + \mathfrak{n} \tag{2.48}$$

where $\Lambda_j^{\langle i\rangle}(u)$ have the form (2.46).

In a spin chain context, the number L of evaluation representations is the number of sites of the chain, the weights $\lambda^{\langle i\rangle} = (\lambda_1^{\langle i\rangle}, \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}}^{\langle i\rangle})$, $i = 1, \ldots, L$ characterize the $\mathcal{B}$ representation (the spin) on each of these sites, and the evaluation parameter a_i is the so-called inhomogeneity parameter at site i.

From the mathematical point of view, evaluation representations are relevant because of:

Theorem 2.3 *All finite dimensional irreducible representations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ can be constructed as (subquotient of) tensor products of evaluation representations.*

This theorem is proven in [44, 45] for $\mathcal{Y}(\mathfrak{n})$ (see also [46, 47]). It is proven in [48, 49, 50, 46] for $\widehat{\mathcal{U}}_q(gl(\mathfrak{n}))$ and in [41] for $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ (see also [51]). We don't know any reference for the case of $\widehat{\mathcal{U}}_q(gl(\mathfrak{m}|\mathfrak{n}))$, but the proof should be similar to the other cases, and, at least, one can construct a wide set of finite dimensional irreducible representations from tensor product of evaluation representations.

Hence, the study of spin chains amounts to study finite dimensional representations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, and the nested Bethe ansatz can be viewed as the construction of a Gelfand-Tsetlin type basis.

2.5 Case of indecomposable (superalgebras) representations

It is well-known that (most of the) Lie superalgebras (and specifically the $gl(\mathfrak{m}|\mathfrak{n})$ superalgebras studied in the present paper) contain finite dimensional representations which are indecomposable. To discuss these special cases, we first remind some definitions about representations of Lie (super)algebras (see e.g. [52] for more details).

2.5.1 Definitions

We focus on finite dimensional representations.

Definition 2.4 *A representation is called **irreducible** if it does not contain any non-trivial invariant subspace. A representation which is not irreducible is called **reducible**.*

Definition 2.5 *A representation is called **fully reducible** if, for any invariant subspace, there exists a complementary subspace which is also invariant. A reducible representation which is not fully reducible is called **indecomposable**.*

It may be useful to illustrate these various definitions. If one considers finite dimensional representations, the representation of the Lie (super)algebra generators are square matrices.

Considering a general linear combination of all these matrices, we have roughly the following (very sketchy) picture

$$\pi_V(\mathcal{A}) = \begin{pmatrix} * & \dots & * \\ \vdots & * & \vdots \\ * & \dots & * \end{pmatrix} \qquad V \text{ irreducible} \qquad (2.49)$$

$$\pi_V(\mathcal{A}) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \qquad V \text{ fully reducible } (V = V_1 \oplus V_2) \qquad (2.50)$$

$$\pi_V(\mathcal{A}) = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \qquad V \text{ indecomposable} \qquad (2.51)$$

where $*$ denotes the non-zero entries and V is the representation of $\mathcal{A}$ under consideration.

Theorem 2.6 *Any finite dimensional representation of semi-simple Lie algebras is fully reducible. This is not the case of $gl(\mathfrak{m}|\mathfrak{n})$ superalgebras. In particular, for these superalgebras, the tensor product of irreducible representations is not always fully reducible.*

Examples of indecomposable representations and of indecomposable tensor products can be found for $sl(1|2)$ in e.g. [53, 54].

Definition 2.7 *A vector v in a vector space V, representation of $\mathcal{A}$, is called **cyclic** if the (iterative) action of all the generators of $\mathcal{A}$ on v span all V.*

For example, in an irreducible highest weight representation, the highest weight vector is cyclic. This is not true for a fully reducible highest weight representation, which contains several highest weight vectors, all of them being needed to span the full representation. However, in an indecomposable representation (finite dimensional), there exists a highest weight vector which is cyclic (see example in [53]).

2.5.2 Application to spin super-chains

It is natural to wonder whether the presence of indecomposable representations on a spin chain alters the Bethe ansatz technics. We argue (and prove in some cases) that the algebraic Bethe ansatz still works. The reasonning is done for the $gl(\mathfrak{m}|\mathfrak{n})$ superalgebras, but quite likely it applies to the deformed case.

The basic ingredient for the algebraic Bethe ansatz is the existence of a cyclic highest weight vector. Through the recursive application of creation-like generators (see section below), one constructs the eigenvectors of the transfer matrix. Hence, it is not the reducibility of the representations, but rather the existence of highest weight that guarantees the Bethe ansatz technics. Indeed, when the representations are fully reducible, one gets several highest weight vectors. In that case, one needs to apply the ansatz on each of the highest weight vectors, but

the technics is still valid. For indecomposable representations, since there is a cyclic highest weight vector, it is very plausible that the Bethe ansatz works.

In particular, for indecomposable representations obtained from the tensor product of two irreducible representations, one can prove that the Bethe ansatz indeed works in the following way. From the original spin chain (that contains the indecomposable representation(s)), one constructs a new chain, where each of the sites carrying an indecomposable representation is replaced by two sites, one for each irreducible representation underlying the indecomposable one. Obviously, the new chain is equivalent to the original one. Moreover, since the new chain contains only irreducible representations, it is clear that one can apply the algebraic Bethe ansatz, to this chain as well as to the original one.

Since no classification of indecomposable representations is known, it is difficult to tell whether they can all be obtained from tensor products of irreducible ones. Nevertheless, we have argued above that the Bethe ansatz should still work in all cases.

As a last remark, let us add that the algebraic structures underlying spin chains are not the finite dimensional (super)algebras, but rather the infinite dimensional ones (super-Yangians or affine quantum algebras). For these algebras, tensor products of representations are in most of the cases also irreducible. Then, the spin chain as a whole appears as a sole (irreducible) representation of these algebras, although it is reducible for the finite dimensional algebra. Thus, it is natural to expect that indecomposable representations of the finite dimensioanl superalgebras appear as 'usual' representations for the infinite dimensional one.

3   Algebraic Bethe Ansatz for the case $\mathfrak{m}+\mathfrak{n}{=}2$

In this section, we remind the framework of the Algebraic Bethe Ansatz (ABA) [55] introduced in order to compute eigenvalues and eigenvectors of the transfer matrix.

For $\mathfrak{m}+\mathfrak{n}=2$, one can consider two different algebras: $\mathcal{A}_2 \equiv \mathcal{A}_{0|2} \equiv \mathcal{A}_{2|0}$ or $\mathcal{A}_{1|1}$. We write the monodromy matrix in the following matricial form:

$$T(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{pmatrix}, \tag{3.1}$$

and the transfer matrix as $t(u) = str(T(u)) = (-1)^{[1]}t_{11}(u) + (-1)^{[2]}t_{22}(u)$. Let Ω be the pseudo-vacuum state presented in previous section:

$$t_{11}(u)\,\Omega \;=\; \Lambda_1(u)\,\Omega \;\;;\;\; t_{22}(u)\,\Omega = \Lambda_2(u)\,\Omega \;\;;\;\; t_{21}(u)\,\Omega = 0\,. \tag{3.2}$$

Using the ternary (RTT) relation one can find the following relations between the different operators of $\mathcal{A}_2$ or $\mathcal{A}_{1|1}$:

$$t_{12}(u)t_{12}(v) \;=\; \begin{cases} t_{12}(v)t_{12}(u), \text{ for } \mathcal{A}_2 \\[2mm] \mathfrak{h}(u,v)t_{12}(v)t_{12}(u), \text{ for } \mathcal{A}_{1|1} \end{cases} \tag{3.3}$$

$$t_{11}(u)\,t_{12}(v) \;=\; \mathfrak{f}_1(u,v)\,t_{12}(v)\,t_{11}(u) + \mathfrak{g}_1^+(u,v)\,t_{12}(u)\,t_{11}(v) \tag{3.4}$$

$$t_{22}(u)\,t_{12}(v) \;=\; \mathfrak{f}_2(v,u)\,t_{12}(v)\,t_{22}(u) + \mathfrak{g}_2^-(v,u)\,t_{12}(u)\,t_{22}(v) \tag{3.5}$$

where we have used the functions

$$\mathfrak{f}_i(u,v) \;=\; \frac{\mathfrak{a}_i(v,u)}{\mathfrak{b}(v,u)} \quad;\quad \mathfrak{g}_i^\pm(u,v) = \frac{\mathfrak{c}_{i\pm1,i}(u,v)}{\mathfrak{b}(u,v)} \quad;\quad \mathfrak{h}(u,v) = (-1)^{[1]+[2]}\frac{\mathfrak{a}_2(u,v)}{\mathfrak{a}_1(u,v)} \tag{3.6}$$

Applying M creation operators we generate a Bethe vector:

$$\Phi(\{u\}) = t_{12}(u_1)\ldots t_{12}(u_M)\,\Omega. \tag{3.7}$$

Demanding $\Phi(\{u\})$ to be an eigenvector of $t(u)$ leads to a set of algebraic relations on the parameters $u_1,\ldots,u_M$, the so-called Bethe equations.

The relation between creation operators prove the invariance (up to a function for $\mathcal{A}_{1|1}$) of the Bethe vector under reordering of the operators $t_{12}(u_j)$. This condition is usefull to compute the unwanted terms from the action of $t(u)$ on $\Phi(\{u\})$. First, we compute the action of $t_{11}(u)$ on $\Phi(\{u\})$:

$$t_{11}(u)\,t_{12}(u_1)\ldots t_{12}(u_M)\,\Omega \;=\; \prod_{k=1}^{M}\mathfrak{f}_1(u,u_k)\,\Lambda_1(u)\,t_{12}(u_1)\ldots t_{12}(u_M)\,\Omega$$

$$+ \sum_{k=1}^{M} P_k(u;\{u_j\})\,t_{12}(u_1)\ldots t_{12}(u_k \to u)\ldots t_{12}(u_M)\,\Omega \tag{3.8}$$

$$P_k(u;\{u_j\}) \;=\; \mathfrak{g}_1^+(u,u_k)\prod_{j\neq k}^{M}\mathfrak{f}_1(u_k,u_j)\,\Lambda_1(u_k)\,. \tag{3.9}$$

where the notation $t_{12}(u_k \to u)$ is used to indicate the position of $t_{12}(u)$ in the ordered product. $P_k(u;\{u_j\})$ corresponds to the (2^M-1) terms containing $\Lambda_1(u_k)$. The form of $P_1(u;\{u_j\})$ is easily computed. The other polynomials $P_k(u;\{u_j\})$ are then computed using the commutation relation between the operators $t_{12}(u_j)$ and puting $t_{12}(u_k)$ on the left. With the same method we compute the action of $t_{22}(u)$ on $\Phi(\{u\})$:

$$t_{22}(u)\,t_{12}(u_1)\ldots t_{12}(u_M)\,\Omega \;=\; \prod_{k=1}^{M}\mathfrak{f}_2(u_k,u)\,\Lambda_2(u)\,t_{12}(u_1)\ldots t_{12}(u_M)\,\Omega$$

$$+ \sum_{k=1}^{M} Q_k(u;\{u_j\})\,t_{12}(u_1)\ldots t_{12}(u_k \to u)\ldots t_{12}(u_M)\,\Omega \tag{3.10}$$

$$Q_k(u;\{u_j\}) \;=\; \mathfrak{g}_2^-(u_k,u)\prod_{j\neq k}^{M}\mathfrak{f}_2(u_j,u_k)\,\Lambda_2(u_k). \tag{3.11}$$

Demanding $\Phi(\{u\})$ to be an eigenvector of $t(u)$ leads to:

$$(-1)^{[1]}P_k(u;\{u\}) + (-1)^{[2]}Q_k(u;\{u\}) = 0, \tag{3.12}$$

which corresponds to the cancelling of the so-called 'unwanted terms' carried by the terms $t_{12}(u_1)\ldots t_{12}(u_k \to u)\ldots t_{12}(u_M)$. In this way, we get the Bethe equations

$$\frac{\Lambda_1(u_k)}{\Lambda_2(u_k)} = \prod_{j\neq k}^{M}\frac{\mathfrak{f}_2(u_k,u_j)}{\mathfrak{f}_1(u_j,u_k)} = (-1)^{M-1}\prod_{j\neq k}^{M}\frac{\mathfrak{a}_2(u_j,u_k)}{\mathfrak{a}_1(u_k,u_j)}\,, \quad k=1,\ldots,M. \tag{3.13}$$

12

Remark that the r.h.s. depends only on the structure constants of the (super)algebra under consideration, while the l.h.s. encodes the representations entering the spin chain.

Then, the eigenvalues of the transfer matrix read

$$t(u)\,\Phi(\{u\}) \;=\; \Big((-1)^{[1]}t_{11}(u) + (-1)^{[2]}t_{22}(u)\Big)\,\Phi(\{u\}) = \Lambda(u;\{u\})\,\Phi(\{u\}) \qquad (3.14)$$

$$\Lambda(u;\{u\}) \;=\; (-1)^{[1]}\Lambda_1(u)\prod_{k=1}^{M}\mathfrak{f}_1(u,u_k) + (-1)^{[2]}\Lambda_2(u)\prod_{k=1}^{M}\mathfrak{f}_2(u_k,u)\,. \qquad (3.15)$$

Note that Bethe equations correspond to the vanishing of the residue of $\Lambda(u;\{u\})$. This is the tool used in analytical Bethe ansatz to obtain Bethe equations, see e.g. [7, 17].

4 Nested Bethe Ansatz

The method, called the Nested Bethe Ansatz (NBA), consists in a recurrent application of the ABA to express higher rank solutions using the lower ones. It has been introduced in [20]. In this way we can compute the eigenvalues, eigenvectors and Bethe equations of the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ model from the ones of $\mathcal{A}_2$ or $\mathcal{A}_{1|1}$ model.

Although we are in a (tensor product of) representation(s) of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, we will loosely keep writing $t_{ij}(u)$ the representation of the operators $t_{ij}(u)$, assuming that the reader will understand that when $t_{ij}(u)$ applies to the highest weight Ω, it is in fact its (matricial) representation that is used.

4.1 Preliminaries

As a starter, we decompose the monodromy matrix in the following form (in the auxiliary space $End(\mathbb{C}^{\mathfrak{m}+\mathfrak{n}})$):

$$T(u) = \begin{pmatrix} t_{11}(u) & B^{(1)}(u) \\ C^{(1)}(u) & T^{(2)}(u) \end{pmatrix} \qquad (4.1)$$

where $B^{(1)}(u)$ (resp. $C^{(1)}(u)$) is a row (resp. column) vector of $\mathbb{C}^{\mathfrak{m}+\mathfrak{n}-1}$, and $T^{(2)}(u)$ is a matrix of $End(\mathbb{C}^{\mathfrak{m}+\mathfrak{n}-1})$.

Then, $T^{(2)}(u)$ is itself decomposed in the same way, and more generally, for a given k in $\{1,\ldots,\mathfrak{m}+\mathfrak{n}-2\}$, we gather the generators $t_{kj}(u)$, (resp. $t_{jk}(u)$) $j = k+1,\ldots,\mathfrak{n}+\mathfrak{m}$, in a row (resp. column) vector of $\mathbb{C}^{\mathfrak{m}+\mathfrak{n}-k}$ and $t_{ij}(u)$, $i,j \geq k$, into a matrix of $End(\mathbb{C}^{\mathfrak{m}+\mathfrak{n}-k})$:

$$B^{(k)}(u) \;=\; \sum_{j=k+1}^{\mathfrak{m}+\mathfrak{n}} e_j^t \otimes t_{kj}(u) \quad \text{and} \quad C^{(k)}(u) = \sum_{j=k+1}^{\mathfrak{m}+\mathfrak{n}} e_j \otimes t_{jk}(u) \qquad (4.2)$$

$$T^{(k+1)}(u) \;=\; \sum_{i,j=k+1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t_{ij}(u) \qquad (4.3)$$

$$T^{(k)}(u) \;=\; \begin{pmatrix} t_{kk}(u) & B^{(k)}(u) \\ C^{(k)}(u) & T^{(k+1)}(u) \end{pmatrix}\,. \qquad (4.4)$$

We decompose the transfer matrix in the same way:

$$
\begin{aligned}
t(u) &= t^{(1)}(u) = (-1)^{[1]} t_{11}(u) + t^{(2)}(u)\,, \\
t^{(k)}(u) &= str\left(T^{(k)}(u)\right) = (-1)^{[k]} t_{kk}(u) + t^{(k+1)}(u)\,.
\end{aligned}
\tag{4.5}
$$

At each step of the recursion, the relations between $t^{(k)}(u)$, $T^{(k)}(u)$ and $B^{(k)}(u)$ remain similar:

$$
B_1^{(k)}(u)\, B_2^{(k)}(v) = (-1)^{[k]} \frac{\mathfrak{a}_{k+1}(u,v)}{\mathfrak{a}_k(u,v)} B_2^{(k)}(v)\, B_1^{(k)}(u)\, \mathbb{R}_{12}^{(k+1)}(u,v)
\tag{4.6}
$$

$$
t_{kk}(u)\, B^{(k)}(v) = \mathfrak{f}_k(u,v)\, B^{(k)}(v)\, t_{kk}(u) + \mathfrak{g}_k^+(u,v)\, B^{(k)}(u)\, t_{kk}(v)
\tag{4.7}
$$

$$
T_1^{(k+1)}(u)\, B_2^{(k)}(v) = \mathfrak{f}_{k+1}(v,u)\, B_2^{(k)}(v)\, T_1^{(k+1)}(u)\, \mathbb{R}_{12}^{(k+1)}(u,v)
$$

$$
\qquad\qquad + \mathfrak{g}_{k+1}^-(v,u)\, B_2^{(k)}(u)\, T_1^{(k+1)}(v)\, \mathbb{R}_{12}^{(k+1)}(u,u)
\tag{4.8}
$$

$$
\mathbb{R}_{12}^{(k)}(u,v)\, T_1^{(k)}(u)\, T_2^{(k)}(v) = T_2^{(k)}(v)\, T_1^{(k)}(u)\, \mathbb{R}_{12}^{(k)}(u,v)\,.
\tag{4.9}
$$

These relations are proven using the RTT relations (2.33) and the Yang–Baxter equation (2.7). When $k = \mathfrak{m} + \mathfrak{n} - 1$, one recovers the commutation relations of $\mathcal{A}_2$ or $\mathcal{A}_{1|1}$.

At each step $k = 1, \ldots, \mathfrak{m} + \mathfrak{n} - 1$ of the nesting, we will introduce a family of spectral parameters $u_j^{(k)}$, $j = 1, \ldots, M_k$, the number M_k of these parameters being a free integer. The partial unions of these families will be noted as

$$
\{u^{(\ell)}\} = \bigcup_{k=1}^{\ell} \{u_j^{(k)}, \ j = 1, \ldots, M_k\}
\tag{4.10}
$$

so that the whole family of spectral parameters is $\{u\} = \{u^{(\mathfrak{m}+\mathfrak{n}-1)}\}$.

These parameters correspond to the different pseudo-excitations above the pseudo-vacuum, and the cardinal of $\{u\}$, $M = \sum_{k=1}^{\mathfrak{m}+\mathfrak{n}-1} M_k$, is the total number of these pseudo-excitations. Let us stress that, in the same way the pseudo-vacuum is <u>not</u> the (physical) ground state of the spin chain, these pseudo-excitations (above the pseudo-vacuum) are <u>not</u> physical excitations. However, they do describe states and even it is believed/proven (depending on the cases) that they describe *all* the states of the chain.

4.2 First step of the construction

From the definition of the highest weight, we have

$$
C^{(1)}(u)\,\Omega = 0
\tag{4.11}
$$

and we can use $B^{(1)}(u)$ as a creation operator. However, since $B^{(1)}(u)$ contains only $t_{1j}(u)$ operators, it is clear that we need to act on several vectors to describe the whole representation

with highest weight Ω. The NBA spirit is to construct these different vectors as Bethe vectors of an $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$ chain that is related to the chain we start with.

More generally, at each step k corresponding to the decomposition (4.4) of the monodromy matrix, we use (a suitable refinement of) $B^{(k)}(u)$ as a creation operator acting on a set of (to be defined) vectors. These vectors are constructed as Bethe vectors of an $\mathcal{A}_{\mathfrak{m}-k-1|\mathfrak{n}}$ chain.

At the first step of the recursion, the Bethe vectors have the form:

$$\Phi(\{u\}) \;=\; B_{a_1}^{(1)}(u_1^{(1)})\dots B_{a_{M_1}}^{(1)}(u_{M_1}^{(1)})\,F_{a_1\dots a_{M_1}}^{(1)}(\{u\})\,\Omega \tag{4.12}$$

$$F_{a_1\dots a_{M_1}}^{(1)}(\{u\}) \;\in\; (\mathbb{C}^{\mathfrak{m}-1|\mathfrak{n}})^{\otimes M_1}\otimes \mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}} \tag{4.13}$$

where $F_{a_1\dots a_{M_1}}^{(1)}(\{u\})$ is built from operators $t_{ij}(u)$, $2\le i\le j\le \mathfrak{m}+\mathfrak{n}$ only. Since $B^{(1)}(u)$ belongs to $\mathbb{C}^{\mathfrak{m}-1|\mathfrak{n}}\otimes\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, we have introduced in the construction M_1 additional auxiliary spaces (labelled $a_1,\dots,a_{M_1}$) that are also carried by $F_{a_1\dots a_{M_1}}^{(1)}(\{u\})$. These new auxiliary spaces take care of the linear combination one has to do between the different generators $t_{1j}(u)$, $j=2,\dots,\mathfrak{m}+\mathfrak{n}$, that enter into the construction. In the next step of the recursion, these new auxiliary spaces are re-interpreted as new *quantum* spaces (i.e. new sites) in the fundamental representation of an $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$ chain. We come back on this point later.

Since $F_{a_1\dots a_{M_1}}^{(1)}(\{u\})$ is built up from operators $t_{ij}(u)$, $2\le i\le j\le \mathfrak{m}+\mathfrak{n}$, it obeys the relation (proven in a more general context in lemma 4.2 below)

$$t_{11}(u)\,F_{a_1\dots a_{M_1}}^{(1)}(\{u\})\,\Omega \;=\; \Lambda_1(u)\,F_{a_1\dots a_{M_1}}^{(1)}(\{u\})\,\Omega \tag{4.14}$$

so that the action of $t_{11}(u)$ on $\Phi(\{u\})$ takes the form:

$$t_{11}(u)\,\Phi(\{u\}) \;=\; \Lambda_1(u)\prod_{i=1}^{M_1}\mathfrak{f}_1(u,u_i^{(1)})\,\Phi(\{u\}) + \sum_{j=1}^{M_1}P_j(u;\{u^{(1)}\})\,\Phi_j(\{u\}) \tag{4.15}$$

$$P_j(u;\{u^{(1)}\}) \;=\; \Lambda_1(u_j^{(1)})\,\mathfrak{g}_1^+(u,u_j^{(1)})\prod_{i\neq j}^{M_1}\mathfrak{f}_1(u_i^{(1)},u_j^{(1)}) \tag{4.16}$$

where $\Phi_j(\{u\})$ is deduced from $\Phi(\{u\})$ by the change $u_j^{(1)}\to u$. Expression (4.15) is computed as it has been done in section 3: $P_1(u;\{u^{(1)}\})$ is easy to compute; the other terms are obtained through a reordering of the operators $B^{(1)}(u_j^{(1)})$. For details, see lemma 4.1 below which deals with the general case.

It remains to compute the action of $t^{(2)}(u)$ on $\Phi(\{u\})$. We do it in two stages. We first commute $t^{(2)}(u)$ with the operators $B^{(1)}(u_j^{(1)})$:

$$t^{(2)}(u)\,\Phi(\{u\}) \;=\; \prod_{j=1}^{M_1}\mathfrak{f}_2(u_j^{(1)},u)\,B_{a_1}^{(1)}(u_1^{(1)})\dots B_{a_{M_1}}^{(1)}(u_{M_1}^{(1)})\,\widetilde{t}^{(2)}(u;\{u^{(1)}\})\,F_{a_1\dots a_{M_1}}^{(1)}(\{u\})\,\Omega$$

$$+\sum_{j=1}^{M_1}\widehat{Q}_j(u;\{u^{(1)}\})\,B_{a_1}^{(1)}(u_1^{(1)})\dots B_{a_j}^{(1)}(u)\dots B_{a_{M_1}}^{(1)}(u_{M_1}^{(1)})\,\widetilde{t}^{(2)}(u_j^{(1)};\{u^{(1)}\})\,F_{a_1\dots a_{M_1}}^{(1)}(\{u\})\,\Omega$$

$$\widehat{Q}_j(u;\{u^{(1)}\}) \;=\; \mathfrak{g}_2^-(u_j^{(1)},u)\prod_{i\neq j}^{M_1}\mathfrak{f}_2(u_j^{(1)},u_i^{(1)}) \tag{4.17}$$

where we used the notation

$$\widetilde{t}^{(2)}(u; \{u^{(1)}\}) \;=\; str_a \Big(T_a^{(2)}(u) \overset{\overleftarrow{M_1}}{\underset{j=1}{\prod}} \mathbb{R}_{a,a_j}^{(2)}(u, u_j^{(1)}) \Big) . \tag{4.18}$$

Again, calculation is done for $\widehat{Q}_1(u; \{u^{(1)}\})$ and then generalized to $\widehat{Q}_j(u; \{u^{(1)}\})$ using the reordering lemma 4.1 and the Yang-Baxter equation.

As already mentionned, the calculation makes appear a new transfer matrix $\widetilde{t}^{(2)}(u; \{u^{(1)}\})$ corresponding to an $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$ chain with $L + M_1$ sites, the M_1 additional sites corresponding to fundamental representations of $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$. This interpretation is supported by the relations

$$R_{ab}^{(2)}(u, v) \, \widetilde{T}_a^{(2)}(u; \{u^{(1)}\}) \, \widetilde{T}_b^{(2)}(v; \{u^{(1)}\}) \;=\; \widetilde{T}_b^{(2)}(v; \{u^{(1)}\}) \, \widetilde{T}_a^{(2)}(u; \{u^{(1)}\}) \, R_{ab}^{(2)}(u, v) \tag{4.19}$$

$$\widetilde{T}_a^{(2)}(u; \{u^{(1)}\}) \;=\; T_a^{(2)}(u) \overset{\overleftarrow{M_1}}{\underset{j=1}{\prod}} \mathbb{R}_{aa_j}^{(2)}(u, u_j^{(1)}) \tag{4.20}$$

which ensure that $\widetilde{T}_a^{(2)}(u; \{u^{(1)}\})$ generates $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$, and that $\widetilde{t}^{(2)}(u; \{u^{(1)}\})$ is indeed a transfer matrix which obeys

$$[\widetilde{t}^{(2)}(u; \{u^{(1)}\}), \widetilde{t}^{(2)}(v; \{u^{(1)}\})] = 0 . \tag{4.21}$$

Then, if we assume that $F_{a_1 \ldots a_{M_1}}^{(1)}(\{u\}) \, \Omega$ is an eigenvector of this new transfer matrix:

$$\widetilde{t}^{(2)}(u; \{u^{(1)}\}) \, F_{a_1 \ldots a_{M_1}}^{(1)}(\{u\}) \, \Omega \;=\; \widetilde{\Gamma}^{(2)}(u) \, F_{a_1 \ldots a_{M_1}}^{(1)}(\{u\}) \, \Omega , \tag{4.22}$$

we deduce

$$t^{(2)}(u) \, \Phi(\{u\}) \;=\; \widetilde{\Gamma}^{(2)}(u) \prod_{j=1}^{M_1} \mathfrak{f}_2(u_j^{(1)}, u) \, \Phi(\{u\}) + \sum_{j=1}^{M_1} Q_j(u; \{u^{(1)}\}) \, \Phi_j(\{u\}) \tag{4.23}$$

$$Q_j(u; \{u^{(1)}\}) \;=\; \widetilde{\Gamma}^{(2)}(u_j^{(1)}) \, \mathfrak{g}_2^-(u_j^{(1)}, u) \prod_{\substack{i \neq j}}^{M_1} \mathfrak{f}_2(u_j^{(1)}, u_i^{(1)}) . \tag{4.24}$$

Gathering the relations (4.15) and (4.23), we get a first expression of the action of $t(u)$ on $\Phi(\{u\})$. When we cancel in this expression the unwanted terms (carried by $\Phi_j(\{u\})$), we get the first Bethe equation and a first expression of the eigenvalue:

$$(-1)^{[1]} \Lambda_1(u_j^{(1)}) \, \mathfrak{g}_1^+(u, u_j^{(1)}) \prod_{\substack{i \neq j}}^{M_1} \mathfrak{f}_1(u_j^{(1)}, u_i^{(1)}) + \widetilde{\Gamma}^{(2)}(u_j^{(1)}) \, \mathfrak{g}_2^-(u_j^{(1)}, u) \prod_{\substack{i \neq j}}^{M_1} \mathfrak{f}_2(u_i^{(1)}, u_j^{(1)}) = 0 \tag{4.25}$$

$$t(u) \, \Phi(\{u\}) = \Big((-1)^{[1]} \Lambda_1(u) \prod_{j=1}^{M_1} \mathfrak{f}_1(u, u_j^{(1)}) + \widetilde{\Gamma}^{(2)}(u) \prod_{j=1}^{M_1} \mathfrak{f}_2(u_j^{(1)}, u) \Big) \, \Phi(\{u\}) . \tag{4.26}$$

In the above relations, everything is known *but* the eigenvalue $\widetilde{\Gamma}^{(2)}(u)$, introduced in (4.22), and the explicit form of $F_{a_1 \ldots a_{M_1}}^{(1)}(\{u\})$ ensuring that (4.22) is indeed satisfied.

Thus, at the end of this first recursion step, we have 'reduced' the problem of computing an eigenvector $\Phi(\{u\})$ for the transfer matrix $t(u)$ of an $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ chain with L sites to the problem of

16

computing an eigenvector $\Phi^{(1)}(\{u\}) = F^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\,\Omega$ for the transfer matrix $\widetilde{t}^{(2)}(u;\{u^{(1)}\})$ of an $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$ chain with $L + M_1$ sites.

To prepare the second step, it remains to single out the highest weights corresponding to the fundamental representations carried by the new sites. This is done in the following way.

$$\Phi^{(1)}(\{u\}) = F^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\,\Omega$$

$$\Phi^{(1)}(\{u\}) = \widetilde{B}^{(2)}_{a_1^2}(u_1^{(2)};\{u^{(1)}\}) \ldots \widetilde{B}^{(2)}_{a_{M_2}^2}(u_{M_2}^{(2)};\{u^{(1)}\})\, F^{(2)}_{a_1^2 \ldots a_{M^2}^2}(\{u\})\,\Omega^{(2)} \tag{4.27}$$

$$\Omega^{(2)} = (e_1^{(1)})^{\otimes M_1} \otimes \Omega\,, \tag{4.28}$$

where $e_1^{(1)} = (1, 0, \ldots, 0)^t \in \mathbb{C}^{\mathfrak{m}-1|\mathfrak{n}}$ and $F^{(2)}_{a_1^2 \ldots a_{M_2}^2}(\{u\})$ is built on operators $\widetilde{t}_{ij}(u;\{u^{(1)}\})$, with $j \geq i > 2$. The operators $\widetilde{B}^{(2)}(u;\{u^{(1)}\})$ play the role, for the $\mathcal{A}_{\mathfrak{m}-1|\mathfrak{n}}$ chain of length $L + M_1$, of the operators $B^{(1)}(u)$ for the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ chain of length L. Explicitly, they are obtained from the following decomposition of the monodromy matrix:

$$\widetilde{T}^{(2)}(u;\{u^{(1)}\}) = \begin{pmatrix} \widetilde{t}_{22}(u;\{u^{(1)}\}) & \widetilde{B}^{(2)}(u;\{u^{(1)}\}) \\ \widetilde{C}^{(2)}(u;\{u^{(1)}\}) & T^{(3)}(u;\{u^{(1)}\}) \end{pmatrix} \tag{4.29}$$

where $\widetilde{T}^{(2)}(u;\{u^{(1)}\})$ has been defined in (4.20). Note that if we follow the second step up to the end, we will produce, as in the first step, a new monodromy matrix

$$\widetilde{T}^{(3)}_a(u;\{u^{(2)}\}) = T^{(3)}_a(u;\{u^{(1)}\}) \overset{\overleftarrow{M_2}}{\prod_{j=1}} \mathbb{R}^{(3)}_{aa_j}(u, u_j^{(2)}) \tag{4.30}$$

corresponding to a new chain based on $\mathcal{A}_{\mathfrak{m}-2|\mathfrak{n}}$ and of length $L + M_1 + M_2$. We want to stress the difference between the monodromy matrix $T^{(3)}(u;\{u^{(1)}\})$ appearing at the begining of the second step and the monodromy matrix $\widetilde{T}^{(3)}_a(u;\{u^{(2)}\})$ constructed at the end of the same step.

4.3 General construction at step k

More generally, the step k starts with the problem

$$\widetilde{t}^{(k)}(u;\{u^{(k-1)}\})\,\Phi^{(k-1)}(\{u\}) = \widetilde{\Gamma}^{(k)}(u)\,\Phi^{(k-1)}(\{u\}) \tag{4.31}$$

where

$$\widetilde{t}^{(k)}(u;\{u^{(k-1)}\}) = str\left(\widetilde{T}^{(k)}(u;\{u^{(k-1)}\})\right). \tag{4.32}$$

is the transfer matrix of a $\mathcal{A}_{\mathfrak{m}+1-k|\mathfrak{n}}$ spin chain of length $L + \sum_{j=1}^{k-1} M_j$ (obtained from the previous step). We define

$$\Phi^{(k-1)}(\{u\}) = F^{(k-1)}_{a_1^{k-1} \ldots a_{M_k}^{k-1}}(\{u\})\,\Omega^{(k-1)} = \mathbb{B}^{(k)}(\{u^{(k)}\})\, F^{(k)}_{a_1^k \ldots a_{M_k}^k}(\{u\})\,\Omega^{(k)} \tag{4.33}$$

$$\Omega^{(k)} = (e_1^{(k-1)})^{\otimes M_{k-1}} \otimes \Omega^{(k-1)}\,, \tag{4.34}$$

17

with $e_1^{(k)} = (1, 0, \ldots, 0)^t \in \mathbb{C}^{\mathfrak{m}-k|\mathfrak{n}}$. We have introduced

$$\mathbb{B}^{(k)}(\{u^{(k)}\}) \;\; = \;\; \widetilde{B}_{a_1^k}^{(k)}(u_1^{(k)}; \{u^{(k-1)}\}) \ldots \widetilde{B}_{a_{M_k}^k}^{(k)}(u_{M_k}^{(k)}; \{u^{(k-1)}\}) \tag{4.35}$$

where the operators are extracted from the monodromy matrix:

$$\widetilde{T}^{(k)}(u; \{u^{(k-1)}\}) \;\; = \;\; \begin{pmatrix} \widetilde{t}_{kk}(u; \{u^{(k-1)}\}) & \widetilde{B}^{(k)}(u; \{u^{(k-1)}\}) \\ \widetilde{C}^{(k)}(u; \{u^{(k-1)}\}) & T^{(k+1)}(u; \{u^{(k-1)}\}) \end{pmatrix}. \tag{4.36}$$

Remark 4.1 *In (4.35), we have indicated only the auxiliary spaces a_j^k, $j = 1, \ldots, M_k$. In fact, since $\widetilde{T}^{(k)}$ is viewed as the monodromy matrix of a spin chain of length $L + \sum_{j=1}^{k-1} M_j$, the other spaces a_j^ℓ, $j = 1, \ldots, M_\ell$, $\ell < k$, are now quantum spaces. Thus, they do not appear explicitly in $\widetilde{T}^{(k)}$, as the sites of the original spin chain, but obviously this monodromy matrix (and its components) does depend on all these spaces.*

We extract from $\widetilde{t}^{(k)}(u; \{u^{(k-1)}\})$ the component $\widetilde{t}_{kk}(u; \{u^{(k-1)}\})$,

$$\widetilde{t}^{(k)}(u; \{u^{(k-1)}\}) = (-1)^{[k]} \widetilde{t}_{kk}(u; \{u^{(k-1)}\}) + str\left(\widetilde{T}^{(k+1)}(u; \{u^{(k)}\})\right), \tag{4.37}$$

and compute its action on the vector $\Phi^{(k-1)}(\{u\})$.

At the first stage, we commute $\widetilde{t}_{kk}(u; \{u^{(k-1)}\})$ with the operators $B^{(k)}(u_j^{(k)}; \{u^{(k-1)}\})$:

$$\widetilde{t}_{kk}(u; \{u^{(k-1)}\}) \, \Phi^{(k-1)}(\{u\}) \;\; = \;\; \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_j^{(k)}) \, \mathbb{B}^{(k)}(\{u^{(k)}\}) \, \widetilde{t}_{kk}(u; \{u^{(k-1)}\}) \, \Phi^{(k)}(\{u\})$$

$$+ \sum_{j=1}^{M_k} \widehat{P}_j(u; \{u^{(k-1)}\}) \, \mathbb{B}_j^{(k)}(u; \{u^{(k)}\}) \, \widetilde{t}_{kk}(u_j^{(k)}; \{u^{(k-1)}\}) \, \Phi^{(k)}(\{u\}) \tag{4.38}$$

where we have introduced

$$\mathbb{B}_j^{(k)}(u; \{u^{(k)}\}) \;\; = \;\; \widetilde{B}_{a_1^k}^{(k)}(u_1^{(k)}; \{u^{(k-1)}\}) \ldots B_{a_j^k}^{(k)}(u_j^{(k)} \to u; \{u^{(k-1)}\}) \ldots B_{a_{M_k}^k}^{(k)}(u_{M_k}^{(k)}; \{u^{(k-1)}\})$$

$$\widehat{P}_j(u; \{u^{(k-1)}\}) \;\; = \;\; \mathfrak{g}_k^+(u, u_j^{(k)}) \prod_{i \neq j}^{M_k} \mathfrak{f}_k(u_i^{(k)}, u_j^{(k)}) \tag{4.39}$$

The calculation is done directly for $\widehat{P}_1$ by collecting the terms containing $\widetilde{t}_{kk}(u_1^{(k)}; \{u^{(k-1)}\})$. It is then generalized to $\widehat{P}_j$ thanks to the following reordering lemma:

Lemma 4.1 *For each $k = 1, \ldots, \mathfrak{m} + \mathfrak{n} - 1$ and $j = 1, \ldots, M_k$, we have*

$$\mathbb{B}^{(k)}(\{u^{(k)}\}) \;\; = \;\; \widetilde{B}_j^{(k)}(u_j^{(k)}) \, \widetilde{B}_1^{(k)}(u_1^{(k)}) \ldots \widetilde{B}_{j-1}^{(k)}(u_{j-1}^{(k)}) \, \widetilde{B}_{j+1}^{(k)}(u_{j-1}^{(k)}) \ldots \widetilde{B}_{M_k}^{(k)}(u_{M_k}^{(k)})$$

$$\times \overrightarrow{\prod_{i=1}^{j-1}} (-1)^{[j]} \frac{\mathfrak{a}_{j+1}(u_i^{(k)}, u_j^{(k)})}{\mathfrak{a}_j(u_i^{(k)}, u_j^{(k)})} \, \mathbb{R}_{ij}^{(k+1)}(u_i^{(k)}, u_j^{(k)}) \tag{4.40}$$

where the dependence in $\{u^{(k-1)}\}$ has been omitted in $\widetilde{B}_p^{(k)}$.

<u>Proof:</u> : Direct calculation using the commutation relations (4.6). ∎

Since the new $\mathbb{R}$-matrices appearing in lemma 4.1 commute with $\widetilde{t}_{kk}(u_j^{(k)}; \{u^{(k-1)}\})$, one deduces that all $\widehat{P}_j$ polynomials have the same form.

In a second stage, we compute the action of $\widetilde{t}_{kk}$ on $F^{(k)} \Omega^{(k)}$:

Lemma 4.2 *For $k = 1, 2, \ldots, \mathfrak{m} + \mathfrak{n} - 1$, the vector $F_{a_1 \ldots a_M}^{(k)}(\{u\}) \Omega^{(k)}$ obeys the following relation:*

$$\widetilde{t}_{kk}(u; \{u^{(k-1)}\}) \, F_{a_1 \ldots a_M}^{(k)}(\{u\}) \, \Omega^{(k)} = \widetilde{\Lambda}_k(u; \{u^{(k-1)}\}) \, F_{a_1 \ldots a_M}^{(k)}(\{u\}) \, \Omega^{(k)} \tag{4.41}$$

where $\widetilde{\Lambda}_k(u; \{u^{(k-1)}\})$ is the weight of the representation with highest weight vector $\Omega^{(k)}$:

$$\widetilde{t}_{kk}(u; \{u^{(k-1)}\}) \, \Omega^{(k)} = \widetilde{\Lambda}_k(u; \{u^{(k-1)}\}) \, \Omega^{(k)} \tag{4.42}$$

<u>Proof:</u> : For $k < i, j, l$, the commutation relations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ rewrite:

$$
\begin{aligned}
t_{kk}(u) \, t_{ij}(v) &= t_{ij}(v) \, t_{kk}(u) + (-1)^{([k]+[j])([i]+[j])} \frac{\mathfrak{c}_{ik}(v, u)}{\mathfrak{b}(v, u)} \, t_{kj}(v) \, t_{ik}(u) \\
&\quad - (-1)^{([k]+[i])([k]+[j])} \frac{\mathfrak{c}_{kj}(v, u)}{\mathfrak{b}(v, u)} \, t_{kj}(u) \, t_{ik}(v) \tag{4.43} \\
t_{lk}(u) \, t_{ij}(v) &= (-1)^{([i]+[j])([l]+[k])} \left(1 + \delta_{ik} \frac{\mathfrak{a}_i(v, u) - \mathfrak{b}(v, u)}{\mathfrak{b}(v, u)} \right) t_{ij}(v) \, t_{lk}(u) \\
&\quad + (1 - \delta_{il}) \frac{\mathfrak{c}_{il}(v, u)}{\mathfrak{b}(v, u)} (-1)^{([i]+[j])([l]+[k])} t_{lj}(v) \, t_{ik}(u) \\
&\quad - (-1)^{([i]+[j])([k]+[j])} \frac{\mathfrak{c}_{kj}(v, u)}{\mathfrak{b}(v, u)} \, t_{lj}(u) \, t_{ik}(v) \tag{4.44}
\end{aligned}
$$

Since $F^{(k)}$ contains terms of type $t_{ij}(u)$ with $k < i \leq j$ only, and because of the property

$$\widetilde{t}_{ik}(u; \{u^{(k-1)}\}) \, \Omega^{(k)} = 0, \quad i > k \tag{4.45}$$

we conclude that $\widetilde{t}_{kk}(u; \{u\})$ commutes with $F^{(k)}$.

The action of $\widetilde{t}_{kk}(u; \{u\})$ on $\Omega^{(k)}$ leads to the result. ∎

Gathering equation (4.38) and lemma 4.2, we get the action of $\widetilde{t}_{kk}$ on $\Phi^{(k-1)}(\{u\})$:

$$
\begin{aligned}
\widetilde{t}_{kk}(u; \{u^{(k-1)}\}) \, \Phi^{(k-1)}(\{u\}) &= \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_j^{(k)}) \, \widetilde{\Lambda}_k(u; \{u^{(k-1)}\}) \, \Phi^{(k-1)}(\{u\}) \\
&\quad + \sum_{j=1}^{M_k} P_j(u; \{u^{(k-1)}\}) \, \mathbb{B}_j^{(k)}(u; \{u^{(k)}\}) \, \Phi^{(k)}(\{u\}) \tag{4.46} \\
\Phi^{(k)}(\{u\}) &= F_{a_1^{(k)} \ldots a_{M^k}^k}^{(k)}(\{u\}) \, \Omega^{(k)} \\
P_j(u; \{u^{(k-1)}\}) &= \widetilde{\Lambda}_k(u_j^{(k)}; \{u^{(k-1)}\}) \, \mathfrak{g}_k^+(u, u_j^{(k)}) \prod_{i \neq j}^{M_k} \mathfrak{f}_k(u_i^{(k)}, u_j^{(k)}). \tag{4.47}
\end{aligned}
$$

19

It remains to do the same for $t^{(k+1)}(u; \{u^{(k-1)}\}) = str\left(T^{(k+1)}(u; \{u^{(k-1)}\})\right)$. We first commute $t^{(k+1)}(u; \{u^{(k-1)}\})$ with $\mathbb{B}^{(k)}(\{u^{(k)}\})$ using relations (2.7) and (4.8):

$$t^{(k+1)}(u; \{u^{(k-1)}\})\, \Phi^{(k-1)}(\{u\}) = \prod_{j=1}^{M_k} \mathfrak{f}_{k+1}(u_j^{(k)}, u)\, \mathbb{B}^{(k)}(\{u^{(k)}\})\, \widetilde{t}^{(k+1)}(u; \{u^{(k)}\})\, \Phi^{(k)}(\{u\})$$

$$+ \sum_{j=1}^{M_k} \widehat{Q}_j(u; \{u^{(k)}\})\, \mathbb{B}_j^{(k)}(u; \{u^{(k)}\})\, \widetilde{t}^{(k+1)}(u_j^{(k)}; \{u^{(k)}\})\, \Phi^{(k)}(\{u\}) \tag{4.48}$$

$$\widehat{Q}_j(u; \{u^{(k)}\}) = \mathfrak{g}_{k+1}^-(u_j^{(k)}, u) \prod_{i \neq j}^{M_k} \mathfrak{f}_{k+1}(u_j^{(k)}, u_i^{(k)}). \tag{4.49}$$

It makes appear new monodromy and transfer matrices:

$$\widetilde{T}_a^{(k+1)}(u; \{u^{(k)}\}) = T_a^{(k+1)}(u; \{u^{(k-1)}\}) \prod_{j=1}^{\overleftarrow{M_k}} \mathbb{R}_{a,a_j}^{(k+1)}(u, u_j^{(k)}) \tag{4.50}$$

$$\widetilde{t}^{(k+1)}(u; \{u^{(k)}\}) = str_a\left(\widetilde{T}_a^{(k+1)}(u; \{u^{(k)}\})\right). \tag{4.51}$$

The new monodromy matrix also satisfies the RTT relation

$$R_{ab}^{(k+1)}(u,v)\, \widetilde{T}_a^{(k+1)}(u; \{u^{(k)}\})\, \widetilde{T}_b^{(k+1)}(v; \{u^{(k)}\}) = \widetilde{T}_b^{(k+1)}(v; \{u^{(k)}\})\, \widetilde{T}_a^{(k+1)}(u; \{u^{(k)}\})\, R_{ab}^{(k+1)}(u,v)$$

so that the problem

$$\widetilde{t}^{(k+1)}(u; \{u^{(k)}\})\, \Phi^{(k)}(\{u\}) = \widetilde{\Gamma}^{(k+1)}(u)\, \Phi^{(k)}(\{u\}) \tag{4.52}$$

is integrable, and defines a $\mathcal{A}_{\mathfrak{m}-k|\mathfrak{n}}$ spin chain, with $L + \sum_{j=1}^{k} M_j$ sites.

Assuming the form (4.52), we get

$$t^{(k+1)}(u; \{u^{(k-1)}\})\, \Phi^{(k-1)}(\{u\}) = \widetilde{\Gamma}^{(k+1)}(u) \prod_{j=1}^{M_k} \mathfrak{f}_{k+1}(u_j^{(1)}, u)\, \Phi^{(k-1)}(\{u\})$$

$$+ \sum_{j=1}^{M_k} Q_j(u; \{u^{(k)}\})\, \mathbb{B}_j^{(k)}(u; \{u^{(k)}\})\, \Phi^{(k)}(\{u\}) \tag{4.53}$$

$$Q_j(u; \{u^{(k)}\}) = \widetilde{\Gamma}^{(k+1)}(u_j^{(k)})\, \mathfrak{g}_{k+1}^-(u_j^{(k)}, u) \prod_{i \neq j}^{M_k} \mathfrak{f}_{k+1}(u_j^{(k)}, u_i^{(k)}). \tag{4.54}$$

Gathering (4.46) and (4.54), and comparing them with (4.31), we get the kth Bethe equation and an expression for $\widetilde{\Gamma}^{(k)}(u)$:

$$(-1)^{[k]}\, \widetilde{\Lambda}_k(u_j^{(k)}; \{u^{(k)}\})\, \mathfrak{g}_k^+(u, u_j^{(k)}) \prod_{i \neq j}^{M_k} \mathfrak{f}_k(u_j^{(k)}, u_i^{(k)}) + \widetilde{\Gamma}^{(k+1)}(u_j^{(k)})\, \mathfrak{g}_{k+1}^-(u_j^{(k)}, u) \prod_{i \neq j}^{M_k} \mathfrak{f}_{k+1}(u_i^{(k)}, u_j^{(k)}) = 0$$

$$\widetilde{\Gamma}^{(k)}(u) = (-1)^{[k]}\, \widetilde{\Lambda}_k(u; \{u^{(k)}\}) \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_j^{(k)}) + \widetilde{\Gamma}^{(k+1)}(u) \prod_{j=1}^{M_k} \mathfrak{f}_{k+1}(u_j^{(k)}, u). \tag{4.55}$$

4.4 End of the recursion

To end the recursion, we remark that

$$\widetilde{\Gamma}^{(\mathfrak{m}+\mathfrak{n})}(u) = (-1)^{[\mathfrak{m}+\mathfrak{n}]}\, \widetilde{\Lambda}_{\mathfrak{m}+\mathfrak{n}}(u; \{u^{(\mathfrak{m}+\mathfrak{n})}\}) \tag{4.56}$$

so that $\widetilde{\Gamma}$ is expressed in term of $\widetilde{\Lambda}$:

$$
\begin{aligned}
\widetilde{\Gamma}^{(k)}(u) \;=\;& (-1)^{[k]}\, \widetilde{\Lambda}_k(u; \{u^{(k)}\}) \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_j^{(k)}) \\[4pt]
&+\; \sum_{\ell=k+1}^{\mathfrak{m}+\mathfrak{n}-1} (-1)^{[\ell]}\, \widetilde{\Lambda}_\ell(u; \{u^{(\ell)}\}) \left(\prod_{j=1}^{M_\ell} \mathfrak{f}_\ell(u, u_j^{(\ell)}) \right) \left(\prod_{p=k}^{\ell-1} \prod_{j=1}^{M_p} \mathfrak{f}_{p+1}(u_j^{(p)}, u) \right) \\[4pt]
&+\; (-1)^{[\mathfrak{m}+\mathfrak{n}]}\, \widetilde{\Lambda}_{\mathfrak{m}+\mathfrak{n}}(u; \{u\}) \left(\prod_{p=k}^{\mathfrak{m}+\mathfrak{n}-1} \prod_{j=1}^{M_p} \mathfrak{f}_{p+1}(u_j^{(p)}, u) \right).
\end{aligned}
\tag{4.57}
$$

It remains to compute the values $\widetilde{\Lambda}_k(u; \{u^{(k)}\})$. It is done in the following lemma:

Lemma 4.3 *The eigenvalue $\widetilde{\Lambda}_k(u_j^{(k)}; \{u^{(k)}\})$ of $\widetilde{t}_{kk}(u; \{u^{(k-2)}\})$ on $\Omega^{(k-1)}$ is given by*

$$\widetilde{\Lambda}_k(u; \{u^{(k)}\}) = \Lambda_k(u) \prod_{\ell=1}^{k-2} \prod_{j=1}^{M_\ell} \frac{\mathfrak{b}(u, u_j^{(\ell)})}{\mathfrak{a}_{\ell+1}(u, u_j^{(\ell)})} = \Lambda_k(u) \prod_{\ell=1}^{k-2} \prod_{j=1}^{M_\ell} \frac{1}{\mathfrak{f}_{\ell+1}(u_j^{(\ell)}, u)} \qquad k = 1, \dots, \mathfrak{m}+\mathfrak{n}$$

where we have used $t_{kk}(u)\,\Omega = \Lambda_k(u)\,\Omega$ for the original spin chain.

<u>Proof:</u> For $\ell = 1, \dots, \mathfrak{m}+\mathfrak{n}-1$, we compute:

$$
\begin{aligned}
\left(\overset{\overleftarrow{M_\ell}}{\prod_{j=1}} R_{aa_j}^{(\ell+1)}(u, u_j^{(\ell)}) \right) \left(e_{\ell+1} \right)^{\otimes M_\ell} \;=\;& \left(\prod_{j=1}^{M_\ell} \mathfrak{a}_{\ell+1}(u, u_j^{(\ell)}) \right) E_{\ell+1,\ell+1} \otimes \left(e_{\ell+1} \right)^{\otimes M_\ell} \\[4pt]
&+ \left(\prod_{j=1}^{M_\ell} \mathfrak{b}(u, u_j^{(\ell)}) \right) \sum_{s=\ell+2}^{\mathfrak{m}+\mathfrak{n}} E_{ss} \otimes \left(e_{\ell+1} \right)^{\otimes M_\ell} \\[4pt]
&+ \sum_{p=1}^{M_\ell} \left(\prod_{j=p+1}^{M_\ell} \mathfrak{a}_{\ell+1}(u, u_j^{(\ell)}) \right) \left(\sum_{s=\ell+2}^{\mathfrak{m}+\mathfrak{n}} (-1)^{(p-1)[\ell+1]([s]+[\ell+1])}\, \mathfrak{c}_{\ell+1,s}(u, u_p^{(\ell)}) \right) \left(\prod_{j=1}^{p-1} \mathfrak{b}(u, u_j^{(\ell)}) \right) \\[4pt]
& \times E_{\ell+1,s} \otimes \left(e_{\ell+1} \right)^{\otimes(p-1)} \otimes e_s \otimes \left(e_{\ell+1} \right)^{\otimes(M_\ell - p)}
\end{aligned}
\tag{4.58}
$$

where the calculation has been done in $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ with the identification $e_1^{(\ell)} \equiv e_{\ell+1}$.

In the product of such terms, we want to select the term(s) carried by E_{kk} in the auxiliary space (labelled a in equation (4.58)). Since the matrices E_{ij} appearing in (4.58) are all upper triangular, this implies that each term must be carried by a E_{kk} matrix in space a. Denoting

21

by $E_{kk}^{(a)}$ such matrix, one deduces

$$str_a\left(E_{kk}^{(a)}\overset{\overrightarrow{k-1}}{\prod_{\ell=1}}\overset{\overleftarrow{M_\ell}}{\prod_{j=1}}R_{aa_j}^{(\ell+1)}(u,u_j^{(\ell)})\right)(e_k)^{\otimes M_{k-1}}\otimes\cdots\otimes(e_3)^{\otimes M_2}\otimes(e_2)^{\otimes M_1}=$$

$$=\left(\prod_{\ell=1}^{k-2}\prod_{j=1}^{M_\ell}\mathfrak{b}(u,u_j^{(\ell)})\right)\prod_{j=1}^{M_{k-1}}\mathfrak{a}_k(u,u_j^{(k-1)})\,(e_k)^{\otimes M_{k-1}}\otimes\cdots\otimes(e_3)^{\otimes M_2}\otimes(e_2)^{\otimes M_1}\quad(4.59)$$

Remark that we didn't mention the contribution of the original $t_{kk}(u)$: in fact, since Ω is a highest weight, the monodromy matrix $T(u)$ is also upper triangular, so that we need also to select only $E_{kk}^{(a)}$ for this term. As a consequence, the product of R-matrices on its own must be carried by $E_{kk}^{(a)}$.

Finally, from (4.59) and the normalisation (2.26), we get the result. ∎

From the expression given in lemma 4.3, one deduces that:

$$\widetilde{\Gamma}^{(k)}(u;\{u\})\;=\;\left(\prod_{p=1}^{k-1}\prod_{j=1}^{M_p}\frac{1}{\mathfrak{f}_{p+1}(u_j^{(p)},u)}\right)\left\{(-1)^{[k]}\Lambda_k(u)\left(\prod_{j=1}^{M_k}\mathfrak{f}_k(u,u_j^{(k)})\right)\left(\prod_{j=1}^{M_{k-1}}\mathfrak{f}_k(u_j^{(k-1)},u)\right)\right.$$

$$+\sum_{\ell=k+1}^{\mathfrak{m}+\mathfrak{n}-1}(-1)^{[\ell]}\Lambda_\ell(u)\left(\prod_{j=1}^{M_\ell}\mathfrak{f}_\ell(u,u_j^{(\ell)})\right)\left(\prod_{j=1}^{M_{\ell-1}}\mathfrak{f}_\ell(u_j^{(\ell-1)},u)\right)$$

$$+\left.(-1)^{[\mathfrak{m}+\mathfrak{n}]}\Lambda_{\mathfrak{m}+\mathfrak{n}}(u)\left(\prod_{j=1}^{M_{\mathfrak{m}+\mathfrak{n}-1}}\mathfrak{f}_{\mathfrak{m}+\mathfrak{n}}(u_j^{(\mathfrak{m}+\mathfrak{n}-1)},u)\right)\right\}\,.\quad(4.60)$$

Let us note that since $\mathfrak{b}(u,u)=0$, equation (4.60) implies that:

$$\widetilde{\Gamma}_k(u_j^{(\ell)};\{u\})\;=\;0\quad\text{for}\quad j=1,\ldots,M_\ell;\;\ell=1,\ldots,k-2\quad(4.61)$$

$$\widetilde{\Gamma}_k(u_i^{(k-1)};\{u\})\;=\;(-1)^{[k]}\Lambda_k(u)\left(\prod_{j=1}^{M_k}\mathfrak{f}_k(u_i^{(k-1)},u_j^{(k)})\right)\left(\prod_{p=1}^{k-2}\prod_{j=1}^{M_p}\frac{1}{\mathfrak{f}_{p+1}(u_j^{(p)},u_i^{(k-1)})}\right)$$

$$\text{for}\quad i=1,\ldots,M_{k-1}\,.\quad(4.62)$$

4.5   Final form of Bethe vectors, eigenvalues and equations

Using these expressions and the value of $\widetilde{\Lambda}_k(u;\{u^{(k)}\})$ given in lemma 4.3, one can recast the Bethe equation (4.55) in its final form:

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})}\;=\;(-1)^{M_k}\prod_{i=1}^{M_{k-1}}\frac{\mathfrak{a}_k(u_j^{(k)},u_i^{(k-1)})}{\mathfrak{b}(u_j^{(k)},u_i^{(k-1)})}\prod_{i\neq j}^{M_k}\frac{\mathfrak{a}_k(u_i^{(k)},u_j^{(k)})}{\mathfrak{a}_{k+1}(u_j^{(k)},u_i^{(k)})}\prod_{i=1}^{M_{k+1}}\frac{\mathfrak{b}(u_i^{(k+1)},u_j^{(k)})}{\mathfrak{a}_{k+1}(u_i^{(k+1)},u_j^{(k)})}$$

$$j=1,\ldots,M_k,\quad k=1,\ldots,\mathfrak{m}+\mathfrak{n}-1\quad(4.63)$$

with the convention $M_0=M_{\mathfrak{m}+\mathfrak{n}}=0$. Remark that, in the distinguished gradation, one can simplify these equations, see section 6.

The eigenvalue of the transfer matrix is obtained from (4.60), remarking that $\Lambda(u) = \widetilde{\Gamma}^{(1)}(u)$:

$$\Lambda(u) = \sum_{k=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[k]} \Lambda_k(u) \prod_{j=1}^{M_{k-1}} \mathfrak{f}_k(u_j^{(k-1)}, u) \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_j^{(k)}). \tag{4.64}$$

Again, due to the distinguished gradation, one can simplify the expression of $\Lambda(u)$.

The number of parameter families is $\mathfrak{m} + \mathfrak{n} - 1$. The Bethe equations (4.63) ensure that $\Lambda(u)$ is analytical, in accordance with the analytical Bethe ansatz.

The Bethe vectors take the form

$$\Phi(\{u\}) = B_{a_1}^{(1)}(u_1^{(1)}) \cdots B_{a_{M_1}}^{(1)}(u_{M_1}^{(1)}) F_{a_1 \ldots a_{M_1}}^{(1)}(\{u\}) \Omega \tag{4.65}$$

$$= B_{a_1^1}^{(1)}(u_1^{(1)}) \cdots B_{a_{M_1}^1}^{(1)}(u_{M_1}^{(1)}) \widetilde{B}_{a_2^2}^{(2)}(u_1^{(2)}) \cdots \widetilde{B}_{a_{M_2}^2}^{(2)}(u_{M_2}^{(2)}) \cdots \widetilde{B}_{a_M^{\mathfrak{n}+\mathfrak{m}-1}}^{(\mathfrak{n}+\mathfrak{m}-1)}(u_M^{(\mathfrak{n}+\mathfrak{m}-1)}) \Omega^{(\mathfrak{n}+\mathfrak{m}-1)}.$$

We remind the notation $M = \sum_{j=1}^{\mathfrak{n}+\mathfrak{m}-1} M_j$, $\Omega^{(k)} = \left(e_1^{(k-1)}\right)^{\otimes M_{k-1}} \Omega^{(k-1)}$, $\Omega^{(1)} = \Omega$ and the auxiliary spaces are indicated according to remark 4.1.

4.6 Bethe equation in the distinguished gradation

For this grade, the properties

$$\mathfrak{a}_k(u,v) = \mathfrak{a}_1(u,v) \equiv \mathfrak{a}(u,v) \quad \text{for} \quad k \leq \mathfrak{m} \quad \text{and} \quad \mathfrak{a}_k(u,v) = -\mathfrak{a}(v,u) \quad \text{for} \quad k > \mathfrak{m} \tag{4.66}$$

$$\mathfrak{f}_k(u,v) = \mathfrak{f}_1(u,v) \equiv \mathfrak{f}(u,v) \quad \text{for} \quad k \leq \mathfrak{m} \quad \text{and} \quad \mathfrak{f}_k(u,v) = \mathfrak{f}(v,u) \quad \text{for} \quad k > \mathfrak{m} \tag{4.67}$$

allow to simplify the Bethe equations to the following form

$$\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} = -\prod_{i \neq j}^{M_1} \frac{\mathfrak{f}(u_j^{(1)}, u_i^{(1)})}{\mathfrak{f}(u_i^{(1)}, u_j^{(1)})} \prod_{i=1}^{M_2} \left(\mathfrak{f}(u_j^{(1)}, u_i^{(2)})\right)^{-1} \qquad j = 1, \ldots, M_1, \tag{4.68}$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}} \mathfrak{f}(u_i^{(k-1)}, u_j^{(k)}) \prod_{i \neq j}^{M_k} \frac{\mathfrak{f}(u_j^{(k)}, u_i^{(k)})}{\mathfrak{f}(u_i^{(k)}, u_j^{(k)})} \prod_{i=1}^{M_{k+1}} \left(\mathfrak{f}(u_j^{(k)}, u_i^{(k+1)})\right)^{-1}$$

$$j = 1, \ldots, M_k, \quad k = 2, \ldots, \mathfrak{m} - 1 \tag{4.69}$$

$$\frac{\Lambda_{\mathfrak{m}+1}(u_j^{(\mathfrak{m})})}{\Lambda_{\mathfrak{m}}(u_j^{(\mathfrak{m})})} = -\prod_{i=1}^{M_{\mathfrak{m}-1}} \mathfrak{f}(u_i^{(\mathfrak{m}-1)}, u_j^{(\mathfrak{m})}) \prod_{i=1}^{M_{\mathfrak{m}+1}} \left(\mathfrak{f}(u_i^{(\mathfrak{m}+1)}, u_j^{(\mathfrak{m})})\right)^{-1} \qquad j = 1, \ldots, M_{\mathfrak{m}} \tag{4.70}$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}} \mathfrak{f}(u_j^{(k)}, u_i^{(k-1)}) \prod_{i \neq j}^{M_k} \frac{\mathfrak{f}(u_i^{(k)}, u_j^{(k)})}{\mathfrak{f}(u_j^{(k)}, u_i^{(k)})} \prod_{i=1}^{M_{k+1}} \left(\mathfrak{f}(u_i^{(k+1)}, u_j^{(k)})\right)^{-1}$$

$$j = 1, \ldots, M_k, \quad k = \mathfrak{m} + 1, \ldots, \mathfrak{m} + \mathfrak{n} - 2 \tag{4.71}$$

$$\frac{\Lambda_{\mathfrak{m}+\mathfrak{n}}(u_j^{(\mathfrak{m}+\mathfrak{n}-1)})}{\Lambda_{\mathfrak{m}+\mathfrak{n}-1}(u_j^{(\mathfrak{m}+\mathfrak{n}-1)})} = -\prod_{i=1}^{M_{\mathfrak{m}+\mathfrak{n}-2}} \mathfrak{f}(u_j^{(\mathfrak{m}+\mathfrak{n}-1)}, u_i^{(\mathfrak{m}+\mathfrak{n}-2)}) \prod_{i \neq j}^{M_{\mathfrak{m}+\mathfrak{n}-1}} \frac{\mathfrak{f}(u_i^{(\mathfrak{m}+\mathfrak{n}-1)}, u_j^{(\mathfrak{m}+\mathfrak{n}-1)})}{\mathfrak{f}(u_j^{(\mathfrak{m}+\mathfrak{n}-1)}, u_i^{(\mathfrak{m}+\mathfrak{n}-1)})}$$

$$j = 1, \ldots, M_{\mathfrak{m}+\mathfrak{n}-1}. \tag{4.72}$$

The Bethe equations depend on the highest weights $\Lambda_j(u)$ and on a sole function:

$$
f(u,v) = \frac{\mathfrak{a}(v,u)}{\mathfrak{b}(v,u)} = \begin{cases} \dfrac{u-v+\hbar}{u-v} & \text{for super-Yangians} \\[2mm] \dfrac{q^{-1}\,u^2 - q\,v^2}{u^2 - v^2} & \text{for deformed superalgebras} \end{cases} \tag{4.73}
$$

It is also true for the transfer matrix eigenvalue:

$$
\begin{aligned}
\Lambda(u) \;=\;& \Lambda_1(u) \prod_{j=1}^{M_1} f(u, u_j^{(1)}) + \sum_{k=2}^{\mathfrak{m}} \Lambda_k(u) \prod_{j=1}^{M_{k-1}} f(u_j^{(k-1)}, u) \prod_{j=1}^{M_k} f(u, u_j^{(k)}) \\
& - \sum_{k=\mathfrak{m}+1}^{\mathfrak{m}+\mathfrak{n}-1} \Lambda_k(u) \prod_{j=1}^{M_{k-1}} f(u, u_j^{(k-1)}) \prod_{j=1}^{M_k} f(u_j^{(k)}, u) - \Lambda_{\mathfrak{m}+\mathfrak{n}}(u) \prod_{j=1}^{M_{\mathfrak{m}+\mathfrak{n}-1}} f(u, u_j^{(\mathfrak{m}+\mathfrak{n}-1)}).
\end{aligned} \tag{4.74}
$$

4.7 Cartan eigenvalues of Bethe vectors

It was shown in [7, 13, 17] that the transfer matrix $t(u)$ commutes with the Cartan subalgebra of $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$. Hence, Bethe vectors are also eigenvectors of the Cartan generators. We give hereafter their eigenvalues. Let us remark that when $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}} = \mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ (or $\mathcal{Y}(\mathfrak{n})$) the symmetry algebra extends to the whole $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ algebra. We remind that we note $\lambda^{\langle k \rangle} = (\lambda_1^{\langle k \rangle}, \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}}^{\langle k \rangle})$ the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ highest weight at site k.

For super Yangian $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ Cartan generators have the form:

$$
t_{jj}^{(1)} \;=\; -(-1)^{[j]}\,\hbar \sum_{k=1}^{L} \mathbb{I}^{\otimes k-1} \otimes \mathcal{E}_{jj} \otimes \mathbb{I}^{\otimes L-k-1} \tag{4.75}
$$

$$
t_{jj}^{(1)}\,\Phi(\{u\}) \;=\; -(-1)^{[j]}\,\hbar \left(M_{j-1} - M_j + \sum_{k=1}^{L} \lambda_j^{\langle k \rangle} \right) \Phi(\{u\}). \tag{4.76}
$$

For the super quantum affine algebra $\hat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$, the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ Cartan generators are given by:

$$
l_{jj}^{\pm} \;=\; (-1)^{[j]\,L}\,(q^{\pm H_j})^{\otimes L} \equiv (-1)^{[j]\,L}\,q^{\pm h_j} \tag{4.77}
$$

$$
q^{h_j}\,\Phi(\{u\}) \;=\; \left(\prod_{\ell=1}^{L} \eta_\ell \right) q^{(1-2[j])(M_j - M_{j-1}) + \sum_{k=1}^{L} \lambda_j^{\langle k \rangle}}\,\Phi(\{u\}). \tag{4.78}
$$

5 Form of the Bethe vectors

In this section, we make contact with the expressions obtained in [25, 26] for Bethe vectors of $\mathcal{Y}(\mathfrak{n})$ and $\hat{\mathcal{U}}_q(\mathfrak{n})$ chains. Note that the construction there is quite the same, but the proof is rather different. We have chosen to stick to the original NBA formalism with a constructive approach for the Bethe vectors. In this section, we show how to reproduce some of the results given in [25, 26], such as the recursion formula for Bethe vector and the 'trace form' which is the central result of these papers. We also generalise them to the case of superalgebras.

5.1 Recursion formula for Bethe vectors

From expression (4.65), we can extract a recurrent form for the Bethe vectors:

$$\Phi_M^{\mathfrak{n+m}}(\{u\}) \;=\; B_{a_1^1}^{(1)}(u_1^{(1)}) \cdots B_{a_{M_1}^1}^{(1)}(u_{M_1}^{(1)})\, \widehat{\Psi}_{\{u^{(1)}\}}\left(\Phi_{M-M_1}^{\mathfrak{n+m-1}}(\{u^{(>1)}\})\right) \tag{5.1}$$

$$\widehat{\Psi}_{\{u^{(1)}\}} \;=\; v^{(2)} \circ (\psi \otimes \pi_{u_{M_1}^{(1)}} \otimes \cdots \otimes \pi_{u_1^{(1)}}) \circ \Delta^{(M_1)} \tag{5.2}$$

where π_a is the fundamental representation evaluation homomorphism normalized as:

$$\pi_a: \;\; \begin{array}{ccc} \mathcal{A}_{\mathfrak{m|n}} \otimes End(\mathbb{C}^{\mathfrak{m|n}}) & \to & End(\mathbb{C}^{\mathfrak{m|n}}) \otimes End(\mathbb{C}^{\mathfrak{m|n}}) \\ T(u) & \mapsto & \mathbb{R}(u,a) \end{array} \tag{5.3}$$

$v^{(k)}$ is the application of the highest weight vector from the right,

$$v^{(k)}(X\,\Omega) = X\,(e_1^{(k)})^{\otimes M_{k-1}} \otimes \Omega\,, \tag{5.4}$$

and ψ is the embedding of $\mathcal{A}_{\mathfrak{m-1|n}}$ in $\mathcal{A}_{\mathfrak{m|n}}$ given by

$$\psi: \;\; \begin{array}{ccc} \mathcal{A}_{\mathfrak{m-1|n}} & \to & \mathcal{A}_{\mathfrak{m|n}} \\ t_{ij}(u) & \mapsto & t_{i+1,j+1}(u) \end{array} \tag{5.5}$$

If we denote by $[.]_{\mathfrak{m|n}}$ the grading used in the $\mathcal{A}_{\mathfrak{m|n}}$ superalgebra, the embbeding ψ corresponds to the identification $[j]_{\mathfrak{m-1|n}} = [j+1]_{\mathfrak{m|n}}$.

Expression (5.1) has been given in [25, 26] in the case of $\mathcal{Y}(\mathfrak{n})$ and $\widehat{\mathcal{U}}_q(\mathfrak{n})$ chains. It is also valid in the case of $\mathcal{Y}(\mathfrak{m|n})$ and $\widehat{\mathcal{U}}_q(\mathfrak{m|n})$ superalgebras.

5.2 Supertrace formula for Bethe vectors

We can also write the Bethe vector into a supertrace formula and prove the equivalence with the recurrence relation discussed above.

$$\Phi_M^{\mathfrak{n+m}}(\{u\}) \;=\; (-1)^{A_1} str_{1\dots M}\left(T_1(u_1^{(1)})\dots T_M(u_{M_{\mathfrak{n+m-1}}}^{(\mathfrak{n+m-1})})\,\mathbb{R}_{1\dots M}(\{u\})\right.$$
$$\left. \times\, E_{\mathfrak{n+m,n+m-1}}^{\vec{\otimes} M_{\mathfrak{n+m-1}}} \otimes \cdots \otimes E_{21}^{\vec{\otimes} M_1}\right)\Omega \tag{5.6}$$

$$\mathbb{R}_{1\dots M}(\{u\}) \;=\; \prod_{j<k}\overrightarrow{\prod_{l=1}^{M_k}}\overleftarrow{\prod_{i=1}^{M_j}}\mathbb{R}_{a_l^k a_i^j}(u_l^{(k)}, u_i^{(j)})\frac{\mathfrak{a}_1(u_l^{(k)}, u_i^{(j)})}{\mathfrak{a}_k(u_l^{(k)}, u_i^{(j)})} \tag{5.7}$$

$$A_k \;=\; \sum_{i=k}^{\mathfrak{n+m-2}}\frac{M_i(M_i+1)}{2}[i] \tag{5.8}$$

We note $1, \dots, M$ the ordered sequence of auxiliary spaces $a_1^1, \dots, a_{M_1}^1, a_1^2, \dots, a_{M_{\mathfrak{m+n-1}}}^{\mathfrak{m+n-1}}$. When $[i] = 0$, we recognize the expression given in [25, 26] for the Yangian $\mathcal{Y}(\mathfrak{n})$ and for the quantum group $\widehat{\mathcal{U}}_q(\mathfrak{n})$. The above expression is also valid in the case of $\mathcal{Y}(\mathfrak{m|n})$ and $\widehat{\mathcal{U}}_q(\mathfrak{m|n})$ superalgebras.

Equivalence is proven along the following lines. Starting from expression (5.6), we can extract the M_1 auxiliary spaces corresponding to the first step of the nested Bethe ansatz :

$$\Phi_M^{\mathfrak{n+m}}(\{u\}) = (-1)^{\frac{M_1(M_1+1)}{2}[1]} str_{1...M_1}\Big[T_1(u_1^{(1)})\dots T_{M_1}(u_{M_1}^{(1)})(-1)^{A_2}\, str_{M_1+1...M}\Big($$

$$T_{M_1+1}(u_1^{(2)})\dots T_M(u_{M_{\mathfrak{n+m}-1}}^{(\mathfrak{n+m}-1)})\,\mathbb{R}_{1...M}(\{u\})\,E_{\mathfrak{n+m},\mathfrak{n+m}-1}^{\vec{\otimes}M_{\mathfrak{n+m}-1}}\otimes\cdots\otimes E_{32}^{\vec{\otimes}M_2}\Big)$$

$$\otimes E_{21}^{\vec{\otimes}M_1}\Big]\otimes\Omega$$

Using the isomorphism $End(\mathbb{C}^{\mathfrak{m+n}})\sim\mathbb{C}^{\mathfrak{m+n}}\otimes\mathbb{C}^{\mathfrak{m+n}}$, one can rewrite, for any $A(v)$, the supertrace with an E_{21} matrix as

$$str\Big(T(u)\,A(v)\,E_{21}\Big) = \sum_{j=1}^{\mathfrak{m+n}}\Big(e_1^t\otimes e_j^t\otimes t_{1j}(u)\Big)\,A(v)\,\Big(e_1\otimes e_2\otimes 1\Big)\tag{5.9}$$

$$= (-1)^{[1]+[1]\,[A]}\sum_{j=1}^{\mathfrak{m+n}}\Big(e_j^t\otimes t_{1j}(u)\Big)\,A(v)\,\Big(e_2\otimes 1\Big).\tag{5.10}$$

Using formula (5.9) for the auxiliary spaces $1,\dots,M_1$, and remarking that the case $j_a=1$ for $a=1,\dots,M_1$ does not contribute, we obtain:

$$\Phi_M^{\mathfrak{n+m}}(\{u\}) = B_{a_1^1}^{(1)}(u_1^{(1)})\cdots B_{a_{M_1}^1}^{(1)}(u_{M_1}^{(1)})(-1)^{A_2}\,str_{M_1+1...M}\Big(T_{M_1+1}(u_1^{(2)})\dots T_M(u_{M_{\mathfrak{n+m}-1}}^{(\mathfrak{n+m}-1)})$$

$$\times\mathbb{R}_{1...M}(\{u\})\,E_{\mathfrak{n+m},\mathfrak{n+m}-1}^{\vec{\otimes}M_{\mathfrak{n+m}-1}}\otimes\cdots\otimes E_{32}^{\vec{\otimes}M_2}\Big)\Omega^{(2)}\tag{5.11}$$

To end the proof, we make the following mappings:

$$\mathcal{A}_{\mathfrak{m|n}} \to \mathcal{A}_{\mathfrak{m-1|n}}\tag{5.12}$$

$$[i]_{\mathfrak{m|n}} \to [i-1]_{\mathfrak{m-1|n}}\tag{5.13}$$

$$\mathbb{R}_{a_j^i a_k^1}(u_i^{(j)},u_k^{(1)}) \to \pi_{u_k^{(1)}}(T_{a_j^i}(u_i^{(j)}))\tag{5.14}$$

$$E_{j+1,j}\in\mathbb{C}^{\mathfrak{m|n}} \to E_{j,j-1}\in\mathbb{C}^{\mathfrak{m-1|n}}\tag{5.15}$$

they allow to recover the definition of $\widehat{\Psi}_{\{u^{(1)}\}}$ and the form (5.1).

5.3 Orthogonality relation for Bethe vectors

In this part we prove the condition for the orthogonality of the on-shell Bethe vectors (i.e. when Bethe equations are satisfied).

Let $\mathcal{F}$ be the space of all Bethe vectors. We introduce the Shapovalov form [25, 56]:

$$\langle.,.\rangle : \mathcal{F}\otimes\mathcal{F}\to\mathbb{C}\tag{5.16}$$

which obeys the following properties:

$$\langle\Omega,\Omega\rangle = 1,\quad\text{where }\Omega\text{ is the highest weight vector of }gl(\mathfrak{m|n}),\tag{5.17}$$

$$\langle t_{ij}(u)\,\omega_1,\omega_2\rangle = \langle\omega_1,t_{ji}(u)\,\omega_2\rangle\quad\forall\,\omega_1,\omega_2\in\mathcal{F}.\tag{5.18}$$

Proposition 5.1 $\langle \Phi(\{u\}), \Phi(\{v\}) \rangle$ *is different from zero if and only if* $\{u^{(k)}\} = \{v^{(k)}\}$, $\forall k = 1, \ldots, \mathfrak{m} + \mathfrak{n} - 1$, *the sets being not ordered.*

<u>Proof:</u> From the eigenvalues of $t(u)$ computed within the NBA method we have

$$t(w)\,\Phi(\{v\}) = \Lambda(w; \{v\})\,\Phi(\{v\}) \tag{5.19}$$

$$\langle t(w)\Phi(\{u\}),\, \Phi(\{v\})\rangle = \langle \Phi(\{u\}),\, t(w)\Phi(\{v\})\rangle \tag{5.20}$$

$$\Lambda(w; \{u\})\,\langle \Phi(\{u\}),\, \Phi(\{v\})\rangle = \Lambda(w; \{v\})\,\langle \Phi(\{u\}),\, \Phi(\{v\})\rangle \tag{5.21}$$

where $\{u\}$ and $\{v\}$ refer to two different sets of parameters for the Bethe vector. Thus, in order to get $\langle \Phi(\{u\}),\, \Phi(\{v\})\rangle$ different from zero, we must have:

$$\Lambda(w; \{u\}) = \Lambda(w; \{v\}). \tag{5.22}$$

Since this equality must be satisfied for all values of w, and looking at (4.63), we conclude that all the families of Bethe roots must be the same up to a permutation in each family M_i: $\{u_i^{(k)},\ i = 1, \ldots M_k\} = \{v_j^{(k)},\ j = 1, \ldots M_k\}$ for all k. ∎

5.4 Examples of Bethe vectors

Using the definition of the Bethe vector (5.7), it is easy to compute their explicit form in some specific cases. We illustrate it below, but a general expression in term of the generators $t_{ij}(v)$ is still lacking.

Bethe vectors of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{n} + \mathfrak{m} = 2$ and $M_1 = M$. We reproduce here the well-known case obtained with algebraic Bethe ansatz.

$$\Phi_M^2(\{u^{(1)}\}) = (-1)^{M[2]}\, t_{12}(u_1^{(1)})\cdots t_{12}(u_M^{(1)})\,\Omega \tag{5.23}$$

Note that this expression is also valid when $\mathfrak{n} + \mathfrak{m} > 2$, setting $M_1 = M$ and $M_k = 0$, $k > 1$.

Bethe vectors of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{n} + \mathfrak{m} = 3$, $M_1 = M$ and $M_2 = 1$. This case is a generalization of the case $M_1 = M_2 = 1$ done for $\mathcal{Y}(gl_\mathfrak{n})$ and $\mathcal{U}_q(gl_\mathfrak{n})$ in [25, 26].

$$\Phi_{M+1}^3(u_1^{(1)}, \ldots, u_M^{(1)}, u_1^{(2)}) = \tag{5.24}$$

$$(-1)^{\frac{(M+1)M}{2}[1] + M[2]} \left(\prod_{i=1}^{M} \frac{\mathfrak{b}(u_1^{(2)}, u_i^{(1)})}{\mathfrak{a}_2(u_1^{(2)}, u_i^{(1)})} \right) \Big\{ (-1)^{[3]}\, t_{12}(u_1^{(1)})\cdots t_{12}(u_M^{(1)})\, t_{23}(u_1^{(2)})$$

$$+ (-1)^{[2]} \sum_{i=1}^{M} \frac{\mathfrak{c}_{23}(u_1^{(2)}, u_i^{(1)})}{\mathfrak{b}(u_1^{(2)}, u_i^{(1)})} \prod_{k=i+1}^{M} \frac{\mathfrak{a}_2(u_1^{(2)}, u_i^{(1)})}{\mathfrak{b}(u_1^{(2)}, u_i^{(1)})}$$

$$\times t_{12}(u_1^{(1)})\cdots t_{12}(u_{i-1}^{(1)})\, t_{13}(u_i^{(1)})\, t_{12}(u_{i+1}^{(1)})\cdots t_{12}(u_M^{(1)})\, t_{22}(u_1^{(2)}) \Big\}\,\Omega$$

Again, this expression is also valid when $\mathfrak{n} + \mathfrak{m} > 3$, setting $M_k = 0$, $k > 2$.

Bethe vectors of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{n} + \mathfrak{m} = 4$, $M_1 = M$ and $M_2 = M_3 = 1$. This case is a generalization of the case $M_1 = M_2 = M_3 = 1$ done for $\mathcal{Y}(gl_\mathfrak{n})$ and $\mathcal{U}_q(gl_\mathfrak{n})$ in [25, 26].

$$
\Phi^4_{M+2}(u^{(1)}_1, \ldots, u^{(1)}_M, u^{(2)}_1, u^{(3)}_1) = (-1)^{A_1+[2]M_1+[4]} \left(\prod_{j=1}^{M_1} \frac{\mathfrak{b}(u^{(3)}_1, u^{(1)}_j)}{\mathfrak{a}_3(u^{(3)}_1, u^{(1)}_j)} \right) \left(\prod_{j=1}^{M_1} \frac{\mathfrak{b}(u^{(2)}_1, u^{(1)}_j)}{\mathfrak{a}_2(u^{(2)}_1, u^{(1)}_j)} \right)
$$

$$
\times \frac{\mathfrak{b}(u^{(3)}_1, u^{(2)}_1)}{\mathfrak{a}_3(u^{(3)}_1, u^{(2)}_1)} \Big\{ (-1)^{[3]}\, t_{12}(u^{(1)}_1) \ldots t_{12}(u^{(1)}_M) t_{23}(u^{(2)}_1) t_{34}(u^{(3)}_1)\, \Omega
$$

$$
+ (-1)^{[4]} \frac{\mathfrak{c}_{34}(u^{(3)}_1, u^{(2)}_1)}{\mathfrak{b}(u^{(3)}_1, u^{(2)}_1)}\, t_{12}(u^{(1)}_1) \ldots t_{12}(u^{(1)}_M) t_{24}(u^{(2)}_j) t_{33}(u^{(3)}_1)\, \Omega
$$

$$
+ \sum_{k=1}^{M_1} (-1)^{[2]} \frac{\mathfrak{c}_{23}(u^{(2)}_1, u^{(1)}_i)}{\mathfrak{b}(u^{(2)}_1, u^{(1)}_i)} \left(\prod_{j=k+1}^{M_1} \frac{\mathfrak{a}_2(u^{(2)}_1, u^{(1)}_k)}{\mathfrak{b}(u^{(2)}_1, u^{(1)}_k)} \right)
$$

$$
\times t_{12}(u^{(1)}_1) \ldots t_{13}(u^{(1)}_k) \ldots t_{12}(u^{(1)}_M) t_{22}(u^{(2)}_1) t_{34}(u^{(3)}_1)\, \Omega
$$

$$
+ \sum_{k=1}^{M_1} (-1)^{[2]} \frac{\mathfrak{c}_{24}(u^{(2)}_1, u^{(1)}_k)}{\mathfrak{b}(u^{(2)}_1, u^{(1)}_k)} \frac{\mathfrak{c}_{34}(u^{(3)}_1, u^{(2)}_1)}{\mathfrak{b}(u^{(3)}_1, u^{(2)}_1)} \left(\prod_{j=k+1}^{M_1} \frac{\mathfrak{a}_2(u^{(2)}_1, u^{(1)}_j)}{\mathfrak{b}(u^{(2)}_1, u^{(1)}_j)} \right)
$$

$$
\times t_{12}(u^{(1)}_1) \ldots t_{14}(u^{(1)}_k) \ldots t_{12}(u^{(1)}_M) t_{22}(u^{(2)}_1) t_{33}(u^{(3)}_1)\, \Omega \Big\} \tag{5.25}
$$

where A_1 is defined in (5.8).

6 Application to AdS/CFT correspondence

To illustrate the technics, we present some Bethe vectors in the case of $\mathcal{A}_{2|1}$, $\mathcal{A}_{2|2}$ and $\mathcal{A}_{4|4}$. These superalgebras, when they are undeformed, appeared recently in the AdS/CFT correspondance, so that it may be useful to look for their Bethe equations, their Bethe eigenvalues and vectors. To encompass future possible developements, we treat both the deformed and undeformed cases. We focus on distinguished gradation, as dealt in section 4.6. The transfer matrix eigenvalues are then given by (4.74), where the weights $\Lambda_j(u)$ depends on representations at each site. If we focus on fundamental representations on each site, with inhomogeneity parameters a_l, $l = 1, \ldots, L$, they take the form

$$
\Lambda_1(u) = \begin{cases} \prod_{l=1}^{L} (u - a_l - \hbar) \\ \prod_{l=1}^{L} \left(\frac{q\,u}{a_l} - \frac{a_l}{q\,u} \right) \end{cases} \qquad \Lambda_j(u) = \begin{cases} \prod_{l=1}^{L} (u - a_l) \\ \prod_{l=1}^{L} \left(\frac{u}{a_l} - \frac{a_l}{u} \right) \end{cases} \qquad j = 2, \ldots, \mathfrak{m} + \mathfrak{n} - 1 \tag{6.1}
$$

where the first line corresponds to $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ and the second one to $\mathcal{U}_q(\mathfrak{m}|\mathfrak{n})$.

6.1 $\mathcal{A}_{2|1}$ spin chains

In addition to the $\mathcal{A}_2$ Bethe vectors (5.23), one can consider the vectors (5.24) that simplifies as (up to a normalisation coefficient):

$$
\begin{aligned}
\Phi^3_{M+1}(u_1^{(1)},\ldots,u_M^{(1)},u_1^{(2)}) \;=\; & t_{12}(u_1^{(1)})\cdots t_{12}(u_M^{(1)})\,t_{23}(u_1^{(2)})\,\Omega \\
& - \sum_{i=1}^{M} \frac{\mathfrak{c}_{23}(u_1^{(2)},u_i^{(1)})}{\mathfrak{b}(u_1^{(2)},u_i^{(1)})} \left(\prod_{k=i+1}^{M} \mathfrak{f}(u_i^{(1)},u_1^{(2)}) \right) \\
& \times t_{12}(u_1^{(1)})\cdots t_{12}(u_{i-1}^{(1)})\,t_{13}(u_i^{(1)})\,t_{12}(u_{i+1}^{(1)})\cdots t_{12}(u_M^{(1)})\,t_{22}(u_1^{(2)})\,\Omega
\end{aligned}
\tag{6.2}
$$

where the function $\mathfrak{f}(u,v)$ is given in (4.73) and the functions $\mathfrak{b}(u,v)$ and $\mathfrak{c}_{jk}(u,v)$ are given in (2.18)-(2.21). The form of the Bethe equations depend on the superalgebra one chooses:

$gl(2|1)$ spin chain

$$
\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} \;=\; - \prod_{i\neq j}^{M_1} \frac{u_i^{(1)}-u_j^{(1)}-\hbar}{u_i^{(1)}-u_j^{(1)}+\hbar} \prod_{i=1}^{M_2} \frac{u_i^{(2)}-u_j^{(1)}}{u_i^{(2)}-u_j^{(1)}-\hbar} \qquad j=1,\ldots,M_1
\tag{6.3}
$$

$$
\frac{\Lambda_3(u_j^{(2)})}{\Lambda_2(u_j^{(2)})} \;=\; - \prod_{i=1}^{M_1} \frac{u_j^{(2)}-u_i^{(1)}-\hbar}{u_j^{(2)}-u_i^{(1)}} \qquad j=1,\ldots,M_2
\tag{6.4}
$$

$\mathcal{U}_q(gl(2|1))$ spin chain

$$
\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} \;=\; - \prod_{i\neq j}^{M_1} \frac{q\,(u_i^{(1)})^2 - q^{-1}\,(u_j^{(1)})^2}{q^{-1}\,(u_i^{(1)})^2 - q\,(u_j^{(1)})^2} \prod_{i=1}^{M_2} \frac{(u_i^{(2)})^2-(u_j^{(1)})^2}{q\,(u_i^{(2)})^2 - q^{-1}\,(u_j^{(1)})^2} \qquad j=1,\ldots,M_1,
$$

$$
\frac{\Lambda_3(u_j^{(2)})}{\Lambda_2(u_j^{(2)})} \;=\; - \prod_{i=1}^{M_1} \frac{q\,(u_j^{(2)})^2 - q^{-1}\,(u_i^{(1)})^2}{(u_j^{(2)})^2 - (u_i^{(1)})^2} \qquad j=1,\ldots,M_2
$$

6.2 $\mathcal{A}_{2|2}$ spin chain

In addition to the vectors (5.23) and (6.2), the vector (5.25) rewrites (up to normalisation):

$$
\begin{aligned}
\Phi^4_{M+2}(\{u\}) \;=\; & t_{12}(u_1^{(1)})\ldots t_{12}(u_M^{(1)})\,t_{23}(u_1^{(2)})\,t_{34}(u_1^{(3)})\,\Omega \\
& + \frac{\mathfrak{c}_{34}(u_1^{(3)},u_1^{(2)})}{\mathfrak{b}(u_1^{(3)},u_1^{(2)})}\, t_{12}(u_1^{(1)})\ldots t_{12}(u_M^{(1)})\,t_{24}(u_j^{(2)})\,t_{33}(u_1^{(3)})\,\Omega \\
& - \sum_{k=1}^{M_1} \frac{\mathfrak{c}_{23}(u_1^{(2)},u_i^{(1)})}{\mathfrak{b}(u_1^{(2)},u_i^{(1)})} \left(\prod_{j=k+1}^{M_1} \mathfrak{f}(u_j^{(1)},u_1^{(2)}) \right) \\
& \times t_{12}(u_1^{(1)})\ldots t_{12}(u_{k-1}^{(1)})\,t_{13}(u_k^{(1)})\,t_{12}(u_{k+1}^{(1)})\ldots t_{12}(u_M^{(1)})\,t_{22}(u_1^{(2)})\,t_{34}(u_1^{(3)})\,\Omega \\
& - \sum_{k=1}^{M_1} \frac{\mathfrak{c}_{24}(u_1^{(2)},u_k^{(1)})}{\mathfrak{b}(u_1^{(2)},u_k^{(1)})} \frac{\mathfrak{c}_{34}(u_1^{(3)},u_1^{(2)})}{\mathfrak{b}(u_1^{(3)},u_1^{(2)})} \left(\prod_{j=k+1}^{M_1} \mathfrak{f}(u_j^{(1)},u_1^{(2)}) \right) \\
& \times t_{12}(u_1^{(1)})\ldots t_{12}(u_{k-1}^{(1)})\,t_{14}(u_k^{(1)})\,t_{12}(u_{k+1}^{(1)})\ldots t_{12}(u_M^{(1)})\,t_{22}(u_1^{(2)})\,t_{33}(u_1^{(3)})\,\Omega
\end{aligned}
\tag{6.5}
$$

$gl(2|2)$ **spin chain**

$$\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} = -\prod_{i\neq j}^{M_1}\frac{u_i^{(1)}-u_j^{(1)}-\hbar}{u_i^{(1)}-u_j^{(1)}+\hbar}\ \prod_{i=1}^{M_2}\frac{u_i^{(2)}-u_j^{(1)}}{u_i^{(2)}-u_j^{(1)}-\hbar} \qquad j=1,\dots,M_1 \qquad (6.6)$$

$$\frac{\Lambda_3(u_j^{(2)})}{\Lambda_2(u_j^{(2)})} = -\prod_{i=1}^{M_1}\frac{u_j^{(2)}-u_i^{(1)}-\hbar}{u_j^{(2)}-u_i^{(1)}}\ \prod_{i=1}^{M_3}\frac{u_i^{(3)}-u_j^{(2)}}{u_i^{(3)}-u_j^{(2)}+\hbar} \qquad j=1,\dots,M_2 \qquad (6.7)$$

$$\frac{\Lambda_4(u_j^{(3)})}{\Lambda_3(u_j^{(3)})} = -\prod_{i=1}^{M_2}\frac{u_j^{(3)}-u_i^{(2)}-\hbar}{u_j^{(3)}-u_i^{(2)}}\ \prod_{i\neq j}^{M_3}\frac{u_i^{(3)}-u_j^{(3)}+\hbar}{u_i^{(3)}-u_j^{(3)}-\hbar} \qquad j=1,\dots,M_3 \qquad (6.8)$$

$\mathcal{U}_q(gl(2|2))$ **spin chain**

$$\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} = -\prod_{i\neq j}^{M_1}\frac{q\,(u_i^{(1)})^2-q^{-1}\,(u_j^{(1)})^2}{q^{-1}\,(u_i^{(1)})^2-q\,(u_j^{(1)})^2}\ \prod_{i=1}^{M_2}\frac{(u_i^{(2)})^2-(u_j^{(1)})^2}{q\,(u_i^{(2)})^2-q^{-1}\,(u_j^{(1)})^2} \qquad j=1,\dots,M_1\,,$$

$$\frac{\Lambda_3(u_j^{(2)})}{\Lambda_2(u_j^{(2)})} = -\prod_{i=1}^{M_1}\frac{q\,(u_j^{(2)})^2-q^{-1}\,(u_i^{(1)})^2}{(u_j^{(2)})^2-(u_i^{(1)})^2}\ \prod_{i=1}^{M_3}\frac{(u_i^{(3)})^2-(u_j^{(2)})^2}{q^{-1}\,(u_i^{(3)})^2-q\,(u_j^{(2)})^2} \qquad j=1,\dots,M_2$$

$$\frac{\Lambda_4(u_j^{(3)})}{\Lambda_3(u_j^{(3)})} = -\prod_{i=1}^{M_2}\frac{q\,(u_j^{(3)})^2-q^{-1}\,(u_i^{(2)})^2}{(u_j^{(3)})^2-(u_i^{(2)})^2}\ \prod_{i\neq j}^{M_3}\frac{q\,(u_i^{(3)})^2-q^{-1}\,(u_j^{(3)})^2}{q^{-1}\,(u_i^{(3)})^2-q\,(u_j^{(3)})^2} \qquad j=1,\dots,M_3$$

6.3 $\mathcal{A}_{4|4}$ **spin chain**

The form of new Bethe vectors is becoming very complicated and we refrain from giving an example. However, since $\mathcal{A}_{4|4}$ is the first superalgebra in the 'super-Yang-Mills series' $\mathcal{A}_2 \equiv \mathcal{A}_{2|0}$, $\mathcal{A}_{2|1}$, $\mathcal{A}_{2|2}$, $\mathcal{A}_{4|4}$, that leads to generic Bethe ansatz equations, we write them:

$gl(4|4)$ **spin chain**

$$\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} = -\prod_{i\neq j}^{M_1}\frac{u_i^{(1)}-u_j^{(1)}-\hbar}{u_i^{(1)}-u_j^{(1)}+\hbar}\ \prod_{i=1}^{M_2}\frac{u_i^{(2)}-u_j^{(1)}}{u_i^{(2)}-u_j^{(1)}-\hbar} \qquad j=1,\dots,M_1$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}}\frac{u_j^{(k)}-u_i^{(k-1)}-\hbar}{u_j^{(k)}-u_i^{(k-1)}}\ \prod_{i\neq j}^{M_k}\frac{u_i^{(k)}-u_j^{(k)}-\hbar}{u_i^{(k)}-u_j^{(k)}+\hbar}\ \prod_{i=1}^{M_{k+1}}\frac{u_i^{(k+1)}-u_j^{(k)}}{u_i^{(k+1)}-u_j^{(k)}-\hbar}$$
$$j = 1,\dots,M_k\,,\quad k=2,3$$

$$\frac{\Lambda_5(u_j^{(4)})}{\Lambda_4(u_j^{(4)})} = -\prod_{i=1}^{M_3}\frac{u_j^{(4)}-u_i^{(3)}-\hbar}{u_j^{(4)}-u_i^{(3)}}\ \prod_{i=1}^{M_5}\frac{u_i^{(5)}-u_j^{(4)}}{u_i^{(5)}-u_j^{(4)}+\hbar} \qquad j=1,\dots,M_4$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}} \frac{u_j^{(k)} - u_i^{(k-1)} + \hbar}{u_j^{(k)} - u_i^{(k-1)}} \prod_{i\neq j}^{M_k} \frac{u_i^{(k)} - u_j^{(k)} + \hbar}{u_i^{(k)} - u_j^{(k)} - \hbar} \prod_{i=1}^{M_{k+1}} \frac{u_i^{(k+1)} - u_j^{(k)}}{u_i^{(k+1)} - u_j^{(k)} + \hbar}$$

$$j = 1, \ldots, M_k, \quad k = 5, 6$$

$$\frac{\Lambda_8(u_j^{(7)})}{\Lambda_k(u_j^{(7)})} = -\prod_{i=1}^{M_6} \frac{u_j^{(7)} - u_i^{(6)} + \hbar}{u_j^{(7)} - u_i^{(6)}} \prod_{i\neq j}^{M_7} \frac{u_i^{(7)} - u_j^{(7)} + \hbar}{u_i^{(7)} - u_j^{(7)} - \hbar} \quad j = 1, \ldots, M_7,$$

$\mathcal{U}_q(gl(4|4))$ **spin chain**

$$\frac{\Lambda_2(u_j^{(1)})}{\Lambda_1(u_j^{(1)})} = -\prod_{i\neq j}^{M_1} \frac{q\,(u_i^{(1)})^2 - q^{-1}\,(u_j^{(1)})^2}{q^{-1}\,(u_i^{(1)})^2 - q\,(u_j^{(1)})^2} \prod_{i=1}^{M_2} \frac{(u_i^{(2)})^2 - (u_j^{(1)})^2}{q\,(u_i^{(2)})^2 - q^{-1}\,(u_j^{(1)})^2} \quad j = 1, \ldots, M_1,$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}} \frac{q\,(u_j^{(k)})^2 - q^{-1}\,(u_i^{(k-1)})^2}{(u_j^{(k+1)})^2 - (u_i^{(k)})^2} \prod_{i\neq j}^{M_k} \frac{q\,(u_i^{(k)})^2 - q^{-1}\,(u_j^{(k)})^2}{q^{-1}\,(u_i^{(k)})^2 - q\,(u_j^{(k)})^2}$$

$$\times \prod_{i=1}^{M_{k+1}} \frac{(u_i^{(k+1)})^2 - (u_j^{(k)})^2}{q\,(u_i^{(k+1)})^2 - q^{-1}\,(u_j^{(k)})^2} \quad j = 1, \ldots, M_k, \qquad k = 2, 3$$

$$\frac{\Lambda_5(u_j^{(4)})}{\Lambda_4(u_j^{(4)})} = -\prod_{i=1}^{M_3} \frac{q\,(u_j^{(4)})^2 - q^{-1}\,(u_i^{(3)})^2}{(u_j^{(4)})^2 - (u_i^{(3)})^2} \prod_{i=1}^{M_5} \frac{(u_i^{(5)})^2 - (u_j^{(4)})^2}{q^{-1}\,(u_i^{(5)})^2 - q\,(u_j^{(4)})^2} \quad j = 1, \ldots, M_4$$

$$\frac{\Lambda_{k+1}(u_j^{(k)})}{\Lambda_k(u_j^{(k)})} = -\prod_{i=1}^{M_{k-1}} \frac{q^{-1}\,(u_j^{(k)})^2 - q\,(u_i^{(k-1)})^2}{(u_j^{(k)})^2 - (u_i^{(k-1)})^2} \prod_{i\neq j}^{M_k} \frac{q^{-1}\,(u_i^{(k)})^2 - q\,(u_j^{(k)})^2}{q\,(u_i^{(k)})^2 - q^{-1}\,(u_j^{(k)})^2}$$

$$\times \prod_{i=1}^{M_{k+1}} \frac{(u_i^{(k+1)})^2 - (u_j^{(k)})^2}{q^{-1}\,(u_i^{(k+1)})^2 - q\,(u_j^{(k)})^2} \quad j = 1, \ldots, M_k, \quad k = 5, 6$$

$$\frac{\Lambda_8(u_j^{(7)})}{\Lambda_7(u_j^{(7)})} = -\prod_{i=1}^{M_6} \frac{q^{-1}\,(u_j^{(7)})^2 - q\,(u_i^{(6)})^2}{(u_j^{(7)})^2 - (u_i^{(6)})^2} \prod_{i\neq j}^{M_7} \frac{q^{-1}\,(u_i^{(7)})^2 - q\,(u_j^{(6)})^2}{q\,(u_i^{(7)})^2 - q^{-1}\,(u_j^{(6)})^2} \quad j = 1, \ldots, M_7,$$

Acknowledgments

A   Finite dimensional algebras

A.1 $gl(\mathfrak{n})$ and $gl(\mathfrak{m}|\mathfrak{n})$ algebras

The Lie algebra $gl(\mathfrak{n})$ is a vector space over $\mathbb{C}$ spanned by the generators $\{\mathcal{E}_{ij}|i, j = 1, 2, ..., \mathfrak{n}\}$. The bilinear commutator associated to $gl(\mathfrak{n})$ is defined by:

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{kj}\,\mathcal{E}_{il} - \delta_{il}\,\mathcal{E}_{kj}. \tag{A.1}$$

The Lie superalgebra $gl(\mathfrak{m}|\mathfrak{n})$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ spanned by the generators $\{\mathcal{E}_{ij}|i,j = 1, 2, ..., \mathfrak{m} + \mathfrak{n}\}$. The bilinear graded commutator associated to $gl(\mathfrak{m}|\mathfrak{n})$ is defined by:

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}\} = \delta_{kj}\, \mathcal{E}_{il} - (-1)^{([i]+[j])([k]+[l])}\delta_{il}\, \mathcal{E}_{kj}\,. \tag{A.2}$$

It is graded anti-symmetric:

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}\} = -(-1)^{([i]+[j])([k]+[l])}\,[\mathcal{E}_{kl}, \mathcal{E}_{ij}\} \tag{A.3}$$

The highest weight representations of these Lie (super)algebras are characterized by the highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}})$ (with $\mathfrak{m} = 0$ for the non-graded case). Finite dimensional irreducible representations are obtained when the following relations are obeyed:

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\,, \quad i = 1, 2, \ldots, \mathfrak{n} - 1 \quad \text{for} \quad gl(\mathfrak{n}) \tag{A.4}$$

$$\begin{cases} \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\,, & i = 1, 2, \ldots, \mathfrak{m} - 1 \\ \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\,, & i = \mathfrak{m} + 1, \ldots, \mathfrak{n} + \mathfrak{m} - 1 \end{cases} \quad \text{for} \quad gl(\mathfrak{m}|\mathfrak{n}) \tag{A.5}$$

A.2 $\mathcal{U}_q(gl(\mathfrak{n}))$ and $\mathcal{U}_q(gl(\mathfrak{m}|\mathfrak{n}))$ algebra

We suppose that q is not a root of unity.

$\mathcal{U}_q(gl(\mathfrak{n}))$ is an associative algebra over $\mathbb{C}$ generated by $q^{\pm H_j}$, e_i and f_i ($1 \le j \le \mathfrak{n}$, $1 \le i \le \mathfrak{n} - 1$) with the relations:

$$q^{H_i}\, q^{-H_i} = q^{-H_i}\, q^{H_i} = 1 \tag{A.6}$$

$$q^{H_i}\, e_j\, q^{-H_i} = q^{\delta_{ij}-\delta_{ij+1}}\, e_j \tag{A.7}$$

$$q^{H_i}\, f_j\, q^{-H_i} = q^{-\delta_{ij}+\delta_{ij+1}}\, f_j \tag{A.8}$$

$$e_i\, f_j - f_j\, e_i = \delta_{ij}\frac{q^{H_i-H_{i+1}} - q^{-H_i+H_{i+1}}}{q - q^{-1}} \tag{A.9}$$

$$e_i\, e_j = e_j\, e_i \quad , \quad f_i\, f_j = f_j\, f_i \quad \text{for} \quad |i - j| \ge 2 \tag{A.10}$$

$$e_i^2\, e_{i\pm1} - (q - q^{-1})\, e_i\, e_{i\pm1}\, e_i + e_{i\pm1}\, e_i^2 = 0 \quad \text{for} \quad 1 \le i \quad \text{and} \quad i \pm 1 \le \mathfrak{n} \tag{A.11}$$

$$f_i^2\, f_{i\pm1} - (q - q^{-1})\, f_i\, f_{i\pm1}\, f_i + f_{i\pm1}\, f_i^2 = 0 \quad \text{for} \quad 1 \le i \quad \text{and} \quad i \pm 1 \le \mathfrak{n} \tag{A.12}$$

The highest weight finite-dimensional irreducible representations of $\mathcal{U}_q(gl(\mathfrak{n}))$, are characterized by a $gl(\mathfrak{n})$ highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\mathfrak{n}})$ as given in (A.4) and a set of parameters $\eta_j = \pm 1, \pm i$ ($j = 1, \ldots, \mathfrak{n}$), see [42] for more details. Explicitly, the weights of the $\mathcal{U}_q(\mathfrak{n})$ algebra read:

$$(\eta_1\, q^{\lambda_1}, \eta_2\, q^{\lambda_2}, \ldots, \eta_\mathfrak{n}\, q^{\lambda_\mathfrak{n}}) \quad \text{with} \quad \lambda_j - \lambda_{j+1} \in \mathbb{Z}_+ \quad \text{and} \quad \eta_j = \pm 1, \pm i \tag{A.13}$$

$\mathcal{U}_q(gl(\mathfrak{m}|\mathfrak{n}))$ is an associative algebra over $\mathbb{C}$ generated by $q^{\pm H_j}$, e_i and f_i ($1 \le j \le \mathfrak{m} + \mathfrak{n}$, $1 \le i \le \mathfrak{m} + \mathfrak{n} - 1$) with the defining relations:

$$q^{H_i}\, q^{-H_i} = q^{-H_i}\, q^{H_i} = 1 \tag{A.14}$$

$$q^{H_i}\, e_j\, q^{-H_i} = q^{(-1)^{[i]}(\delta_{ij}-\delta_{ij+1})}\, e_j \tag{A.15}$$

$$q^{H_i}\, f_j\, q^{-H_i} = q^{-(-1)^{[i]}(\delta_{ij}-\delta_{ij+1})}\, f_j \tag{A.16}$$

$$e_i \, f_j - (-1)^{([i]+[i+1])([j]+[j+1])} \, f_j \, e_i = \delta_{ij} \frac{q^{H_i - H_{i+1}} - q^{-H_i + H_{i+1}}}{q^{(-1)^{[i]}} - q^{-(-1)^{[i]}}} \tag{A.17}$$

$$e_i \, e_j = (-1)^{([i]+[i+1])([j]+[j+1])} \, e_j \, e_i \quad , \quad f_i \, f_j = (-1)^{([i]+[i+1])([j]+[j+1])} \, f_j \, f_i \tag{A.18}$$

$$e_i^2 \, e_{i\pm1} - (q - q^{-1}) \, e_i \, e_{i\pm1} \, e_i + e_{i\pm1} \, e_i^2 = 0 \quad \text{with} \quad i \neq \mathfrak{m} \tag{A.19}$$

$$f_i^2 \, f_{i\pm1} - (q - q^{-1}) \, f_i \, f_{i\pm1} \, f_i + f_{i\pm1} \, f_i^2 = 0 \quad \text{with} \quad i \neq \mathfrak{m} \tag{A.20}$$

The following identification gives the isomorphism between the RTT presentation and the Serre-Chevalley one [37].

$$l_{ii}^+ = (-1)^{[i]} \, q^{H_i} \quad ; \quad l_{i,i+1}^+ = (-1)^{[i+1]} \, (q - q^{-1}) \, q^{H_i} \, f_i \quad ; \quad l_{i,i-1}^+ = 0 \tag{A.21}$$

$$l_{ii}^- = (-1)^{[i]} \, q^{-H_i} \quad ; \quad l_{i,i-1}^- = (-1)^{[i-1]} \, (q - q^{-1}) \, e_i \, q^{-H_i} \quad ; \quad l_{i,i+1}^- = 0 \tag{A.22}$$

Highest weight finite-dimensional irreducible representations of $\mathcal{U}_q(gl(\mathfrak{m}|\mathfrak{n}))$ have been studied in [43]. They are characterized by a $gl(\mathfrak{m}|\mathfrak{n})$ highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}})$ as given in (A.5) and a set of parameters η_a:

$$((-1)^{[1]}\eta_1 \, q^{\lambda_1}, \ldots, (-1)^{[\mathfrak{m}+\mathfrak{n}]}\eta_{\mathfrak{m}+\mathfrak{n}} \, q^{\lambda_{\mathfrak{m}+\mathfrak{n}}}) \quad \text{with} \quad \begin{cases} \lambda_j - \lambda_{j+1} \in \mathbb{Z}_+ \,, \ j \neq \mathfrak{m} \\ \eta_j = \pm 1 \,, \ j = 1, \ldots, \mathfrak{n} + \mathfrak{m} \end{cases} \tag{A.23}$$

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